Guided Notes

Unit 5: Quadratic Functions

Name:__________________________________________
Teacher:__________________________Per:____

Unit Overview:
In this unit you will study a variety of ways to solve quadratic functions and systems of equations and apply your learning to analyzing real world problems.

Essential Questions:
How are quadratic functions used to model, analyze, and interpret mathematical relationships?
Why is it advantageous to know a variety of ways to solve and graph quadratic functions?

Prerequisite Skills:
• Operations on polynomials
• Factoring polynomials
• Evaluating functions
• Solving equations
• Solving inequalities
• Graphing linear functions
• Interpreting graphs of linear functions
Unit 5 – Getting Ready
1. Determine each product.
   a. \((x - 2)(3x + 5)\)  
   b. \(2y(y + 6)(y - 1)\)

2. Factor each polynomial
   a. \(2x^2 + 14x\)  
   b. \(3x^2 - 75\)  
   c. \(x^2 + 7x + 10\)

3. If \(f(x) = 3x - 5\), find each value
   a. \(f(4)\)  
   b. \(f(-2)\)

4. Solve the equation \(4x - 5 = 19\)  
5. Solve the inequality \(\frac{1}{3}x + 9 > 13\)

6. Explain how to graph \(2x + y = 4\)

The following graph compares calories burned when running and walking at constant rates of 10 mi/h and 2 mi/h, respectively.

7. What does the ordered pair \((1.5, 300)\) represent on this graph?

8. How many calories would be burned after four hours when running and after four hours when walking?
Lesson 17-1: Modeling with a Quadratic Function

Objectives:
- Model a real-world situation with a quadratic function.
- Identify quadratic functions.
- Write a quadratic function in standard form.

NOTES:

1. How is the perimeter of a rectangle determined? How is the area of a rectangle determined?

   \[ P = \quad \text{\_\_\_\_\_\_\_\_\_} \quad \text{\_\_\_\_\_\_\_\_\_} \]
   \[ A = \quad \text{\_\_\_\_\_\_\_\_\_} \quad \text{\_\_\_\_\_\_\_\_\_} \]

2. Complete the table for rectangles with the given side lengths.

3. Express regularity in repeated reasoning. Describe any patterns you observe in the table.

<table>
<thead>
<tr>
<th>Length (yards)</th>
<th>Width (yards)</th>
<th>Perimeter (yards)</th>
<th>Area (square yards)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>150</td>
<td>320</td>
<td>1500</td>
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<tr>
<td>20</td>
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<td>150</td>
<td>320</td>
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</tbody>
</table>

4. Is a 70-yd by 90-yd rectangle the same as a 90-yd by 70-yd rectangle? Explain your reasoning.
5. Graph the data from the table in Item 2 as ordered pairs.
6. Use the table and the graph to explain why the data in Items 2 and 5 are not linear.

7. Describe any patterns you see in the graph.

8. What appears to be the largest area from the data in Items 2 and 5?

9. Write a function $A(l)$ that represents the area of a rectangle whose length is $l$ and whose perimeter is 320.

The function $A(l)$ is called a ________________ because the greatest degree of any term is 2 (an $x^2$ term). The ________________ is $y = ax^2 + bx + c$ or $f(x) = ax^2 + bx + c$, where $a$, $b$, and $c$ are real numbers and $a \neq 0$.

10. Write the function $A(l)$ in standard form. What are the values of $a$, $b$, and $c$?
Lesson 17-1 Homework

Lesson Summary/Reflection:

1. For the function \( f(x) = x^2 + 2x + 3 \), create a table of values for \( x = -3, -2, -1, 0, 1 \). Then sketch a graph of the quadratic function on the grid below.

   ![Graph Grid](image)

2. Barry needs to find the area of a rectangular room with a width that is 2 feet longer than the length. Write an expression for the area of the rectangle in terms of the length.

3. **Critique the reasoning of others.** Sally states that the equation \( g(x) = x^3 + 10x^2 - 3x \) represents a quadratic function. Explain why Sally is incorrect.

4. Write the quadratic function \( f(x) = (3 - x)^2 \) in standard form.
5. Create tables to graph \( y = 3x \) and \( y = 3x^2 \) on the grid below. Explain the differences between the graphs.

\[
\begin{array}{|c|c|}
\hline
x & y \\
\hline
0 & 0 \\
1 & 3 \\
2 & 6 \\
3 & 9 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
x & y \\
\hline
-1 & -3 \\
-2 & -12 \\
-3 & -27 \\
\hline
\end{array}
\]

6. Write each quadratic function in standard form:
   a. \( g(x) = x(x - 2) + 4 \)
   b. \( f(t) = 3 - 2t^2 - t \)

7. **Attend to precision.** Determine whether each function is a quadratic function. Justify your responses.
   a. \( S(r) = 2\pi r^2 + 20\pi r \)
   d. \( f(a) = \frac{a^2 + 4a - 3}{2} \)
   
   b. \( f(x) = 4x^2 - 3x + 2 \)
   e. \( g(x) = 3x^2 + 2x + 1 \)
   
   c. \( f(x) = 4^2x - 3 \)
   f. \( h(x) = \frac{4}{x^2} - 3x + 2 \)
Lesson 17-2: Graphing and Analyzing a Quadratic Function

Objectives:
- Graph a quadratic function.
- Interpret key features of the graph of a quadratic function.

NOTES:
1. Use a graphing calculator to graph \( A(l) \) from Item 9 in Lesson 17-1. Sketch the graph on the grid.

2. Identify the vertex of the graph of \( A(l) \) in Item 1. Does the vertex represent a maximum or a minimum of the function?

3. Examine the graph of \( A(l) \). For what values of the length is the area increasing? For what values of the length is the area decreasing?

4. Describe the point where the area changes from increasing to decreasing.

5. Use the table, the graph, and/or the function to determine the reasonable domain and range of the function \( A(l) \). Describe each using words and an inequality.

\[ \text{The graph of a quadratic function is a curve called a } \]
\[ \text{________________________. A parabola has a point at which a maximum or minimum } \]
\[ \text{value of the function occurs. That point is called the _____________________. The y-value } \]
\[ \text{of the vertex is the __________________ or __________________ of the function.} \]

\[ \text{FIFA regulations state that the length of the touchline of a soccer field must be greater than } \]
\[ \text{the length of the goal line.} \]
6. **Reason abstractly.** Can Coach Wentworth use the rectangle that represents the largest area of \(A(l)\) for her soccer field? Explain why or why not.

**FIFA regulations also state that the length of the touchlines of a soccer field must be at least 100 yds, but no more than 130 yds. The goal lines must be at least 50 yds, but no more than 100 yds.**

7. **Construct viable arguments.** Determine the dimensions of the FIFA regulation soccer field with the largest area and a 320-yd perimeter. Support your reasoning with multiple representations.

8. **Consider the quadratic function** \(f(x) = x^2 - 2x - 3.\)
   a. Write the function in factored form by factoring the polynomial \(x^2 - 2x - 3.\)
   b. To find the \(x\)-intercepts of \(f(x)\), use the factored form of \(f(x)\) and solve the equation \(f(x) = 0.\)
   c. A parabola is symmetric over the vertical line that contains the vertex. How do you think the \(x\)-coordinate of the vertex relates to the \(x\)-coordinates of the \(x\)-intercepts? Use the symmetry of a parabola to support your answer.
   d. Write the vertex of the quadratic function.
Lesson 17-2 Homework

Lesson Summary/Reflection:

1. Complete the table for the quadratic function \( f(x) = -x^2 - 4x - 3 \)

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
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</thead>
<tbody>
<tr>
<td>-4</td>
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<tr>
<td>-3</td>
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<td>-1</td>
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<td>0</td>
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</tbody>
</table>

2. Identify the maximum or minimum value of the quadratic function from #1.

3. Consider the quadratic function \( y = -x^2 + 2x - 3 \).
   a. Create a table of values for the function for domain values 0, 1, 2, 3, and 4.
   b. Sketch a graph of the function. Identify and label the vertex. What is the maximum value of the function?
   c. Use inequalities to write the domain, range, and values of x for which y is decreasing.
4. Write the quadratic function \( g(x) = x(x - 2) + 4 \) in standard form.

Use \( f(x) = x^2 + 6x + 5 \) for #5-8

5. Create a table of values and graph \( f(x) \).

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<tr>
<th>x</th>
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6. Use your graph to identify the maximum or minimum value of \( f(x) \).

7. Write the domain and range of \( f(x) \) using inequalities. D: ___________ R: ___________

8. Determine the values of \( x \) for which \( f(x) \) is increasing. _________________

9. **Make use of structure.** Sketch a graph of a quadratic function with a maximum. Now sketch another graph of a quadratic function with a minimum. Explain the difference between the increasing and decreasing behavior of the two functions.
LESSON 17 PRACTICE
Additional practice problems from lessons 17-1 and 17-2.

Lesson 17-1

1. The base of a rectangular window frame must be 1 foot longer than the height. Which of the following is an equation for the area of the window in terms of the height?
   A. \( A(h) = h + 1 \)    B. \( A(h) = (h + 1)h \)    C. \( A(h) = h^2 - 1 \)    D. \( A(h) = h^2 + h + 1 \)

2. Which of the following is an equation for the area of an isosceles right triangle, in terms of the base?
   
   A. \( A(b) = b^2 \)  
   B. \( A(b) = \frac{1}{2}b^2 \)  
   C. \( A(b) = \frac{\sqrt{b^2}}{2} \)  
   D. \( A(b) = \frac{1}{2}b \)

3. Ben is creating a triangle that has a base that is twice the length of the height.
   a. Write an expression for the base of the triangle in terms of the height.
   b. Write a function for the area, \( A(h) \), of the triangle in terms of the height.
   c. Complete the table and then graph the function.

<table>
<thead>
<tr>
<th>h</th>
<th>A(h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
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<tr>
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<td></td>
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<tr>
<td>3</td>
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<tr>
<td>4</td>
<td></td>
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<tr>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

Use the following information for #4-8:
Jenna is in charge of designing the screen for a new smart phone. The design specs call for a screen that has an outside perimeter of 12 inches.
4. Complete the table for the screen measurements.

<table>
<thead>
<tr>
<th>Width, w</th>
<th>Length, l</th>
<th>Area, A(w)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>5</td>
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<tr>
<td>2</td>
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<tr>
<td>w</td>
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</tbody>
</table>

5. Write a function A(w) for the area of the screen in terms of the width.

6. Graph the function A(w). Label each axis.

7. Determine the domain and range of the function.

8. Determine the maximum area of the screen that Jenna can design. What are the dimensions of this screen?

9. Samantha’s teacher writes the function \( f(x) = 2x^3 - 2x(3 - x + x^2) + 3 \)
   a. Barry tells Samantha that the function cannot be quadratic because it contains the term \( 2x^3 \). What should Barry do to the function before making this assumption?
   b. Is the function a quadratic function? Explain.

10. Identify whether each function is quadratic:
    a. \( y = 2x - 3^2 \)
    b. \( y = 3x^2 - 2x \)
    c. \( y = 2 - \frac{3}{x^2} + x \)
11. State whether the data in each table are linear. Explain why or why not.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
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<tr>
<td>2</td>
<td>-1</td>
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<tr>
<td>3</td>
<td>-4</td>
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<tr>
<td>4</td>
<td>-7</td>
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<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
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</thead>
<tbody>
<tr>
<td>0</td>
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<td>1</td>
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<td>3</td>
<td>2</td>
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<tr>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

**Lesson 17-2**

12. Write each quadratic function in standard form.
   a. \( y = 3x - 5 + x^2 \)  
   b. \( y = 6 - 5x^2 \)  
   c. \( y = -0.5x + \frac{3}{4}x^2 - \pi \)

13. For each function, complete the table of values. Graph the function and identify the maximum or minimum of the function.

   a. \( y = x^2 + 4x - 1 \)

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td></td>
</tr>
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<td>-2</td>
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<td>-1</td>
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<tr>
<td>0</td>
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<tr>
<td>1</td>
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</tbody>
</table>

max/min: __________
b. \[ y = -x^2 + 8x - 13 \]

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td></td>
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<td>3</td>
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<tr>
<td>4</td>
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<td>5</td>
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<td>6</td>
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</table>

max/min: _________


<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
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<tbody>
<tr>
<td>0</td>
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<tr>
<td>-3</td>
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</tbody>
</table>

max/min: _________


### Construct Viable Arguments and Critique the Reasoning of Others

14. As part of her math homework, Kylie is graphing a quadratic function. After plotting several points, she notices that the dependent values are increasing as the independent values increase. She reasons that the function will eventually reach a maximum value and then begin to decrease. Is Kylie correct? Why or why not?
Common Core State Standards for Lesson 18

<table>
<thead>
<tr>
<th>Standard</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>HSA-SSE.A.1</td>
<td>Interpret expressions that represent a quantity in terms of a context.</td>
</tr>
<tr>
<td>HSA-SSE.A.1a</td>
<td>Interpret parts of an expression, such as terms, factors, and coefficients.</td>
</tr>
<tr>
<td>HSA-SSE.A.1b</td>
<td>Interpret complicated expressions by viewing one or more of their parts as a single entity.</td>
</tr>
<tr>
<td>HSA-SSE.B.3</td>
<td>Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.</td>
</tr>
<tr>
<td>HSA-SSE.B.3a</td>
<td>Factor a quadratic expression to reveal the zeros of the function it defines.</td>
</tr>
</tbody>
</table>

Lesson 18-1: Solving by Graphing or Factoring

Objectives:
- Use a graph to solve a quadratic equation.
- Use factoring to solve a quadratic equation.
- Describe the connection between the zeros of a quadratic function and the $x$-intercepts of the function’s graph.

NOTES:

Carter, Alisha, and Joseph are building a trebuchet for an engineering competition. A trebuchet is a medieval siege weapon that uses gravity to launch an object through the air. When the counterweight at one end of the throwing arm drops, the other end rises and a projectile is launched through the air. The path the projectile takes through the air is modeled by a parabola.

To win the competition, the team must build their trebuchet according to the competition specifications to launch a small projectile as far as possible. After conducting experiments that varied the projectile’s mass and launch angle, the team discovered that the ball they were launching followed the path given by the quadratic equation $-\frac{1}{8}x^2 + 2x$.

1. **Make sense of problems.** How far does the ball land from the launching point?

2. What is the maximum height of the ball?

3. What are the $x$-coordinates of the points where the ball is on the ground?

To determine how far the ball lands from the launching point, you can solve the equation $-\frac{1}{8}x^2 + 2x = 0$, because the height $y$ equals 0.

4. Verify that $x = 0$ and $x = 16$ are solutions to this equation.
5. Without the graph, could you have determined these solutions? Explain.

The ____________________________ of a polynomial equation provides an effective way to determine the values of $x$ that make the equation equal 0.

**Zero Product Property**
If $ab = 0$, then either $a = 0$ or $b = 0$.

**Example A** Solve $-\frac{1}{8}x^2 + 2x = 0$ by factoring.

**Try These A** Solve each quadratic equation by factoring.

a) $x^2 - 5x - 14 = 0$  
   b) $3x^2 - 6x = 0$  
   c) $x^2 + 3x = 18$

The graph of the function $y = x^2 - 6x + 5$ is shown.

7. Identify the $x$-intercepts of the graph.

8. What is the $x$-coordinate of the vertex?

9. Describe the $x$-coordinate of the vertex with respect to the two $x$-intercepts.

10. Solve the related quadratic equation $x^2 - 6x + 5 = 0$ by factoring.

11. How do the solutions you found in Item 10 relate to the $x$-intercepts of the above graph?
12. The quadratic function \( y = ax^2 + bx + c \) is related to the equation \( ax^2 + bx + c = 0 \) by letting \( y \) equal zero.

a. Why do you think the \( x \)-coordinates of the \( x \)-intercepts are called the zeros of the function?

b. Describe the relationship between the real roots of a quadratic equation and the zeros of the related quadratic function.
Lesson 18-1 Homework

Lesson Summary/Reflection:

#1-2 Solve by factoring.

1. $x^2 - 81 = 0$  
   2. $\frac{1}{2}x^2 - 2x = 0$

#3-4 Identify the zeros of the quadratic function. How are the linear factors of the quadratic expression related to the zeros?

3. $y = x^2 + 8x + 7$  
   4. $y = x^2 - 3x + 2$

5. **Make use of structure.** Use the graph of the quadratic function $y = 2x^2 + 6x$ shown to determine the roots of the quadratic equation $0 = 2x^2 + 6x$.

Solve by factoring.

6. $x^2 - 2x + 1 = 0$  
   7. $2x^2 - 7x - 4 = 0$  
   8. $x^2 + 5x = 0$
9. Write a quadratic equation whose roots are 3 and −6.

10. A whale jumps vertically from a pool at Ocean World. The function 
    \( y = -16x^2 + 32x \) models the height of a whale in feet above the
    surface of the water after \( x \) seconds.
    
    a) What is the maximum height of the whale above the surface of the water?

    b) How long is the whale out of the water? Justify your answer.

11. **Construct viable arguments.** Is it possible for two different quadratic functions to share the same zeros? Use a graph to justify your response.
Lesson 18-2: The Axis of Symmetry and the Vertex

Objectives:
- Identify the axis of symmetry of the graph of a quadratic function.
- Identify the vertex of the graph of a quadratic function.

NOTES:

The ______________________________ of the parabola determined by the function $y = ax^2 + bx + c$ is the vertical line that passes through the vertex.

The equation for the axis of symmetry is $x = -\frac{b}{2a}$.

The vertex is on the axis of symmetry.

Therefore, the $x$-coordinate of the vertex is $-\frac{b}{2a}$.

Each point on a quadratic graph will have a mirror image point with the same $y$-coordinate that is equidistant from the axis of symmetry. For example, the point $(0, 5)$ is reflected over the axis of symmetry to the point $(6, 5)$ on the graph.

1. Give the coordinates of two additional points that are reflections over the axis of symmetry in the graph above.

2. For each function, identify the zeros graphically. Confirm your answer by setting the function equal to 0 and solving by factoring.

a) $y = x^2 + 2x - 8$

b) $y = -x^2 + 4x + 5$
3. For each graph in Item 2, determine the $x$-coordinate of the vertex by finding the $x$-coordinate exactly in the middle of the two zeros. Confirm your answer by calculating the value of $-\frac{b}{2a}$.
   a) $y = x^2 + 2x - 8$
   b) $y = -x^2 + 4x + 5$

4. For each graph in Item 2, determine the $y$-coordinate of the vertex from the graph. Confirm your answer by evaluating the function at $x = -\frac{b}{2a}$.
   a) $y = x^2 + 2x - 8$
   b) $y = -x^2 + 4x + 5$

5. **Reason quantitatively.** For each graph in Item 2, determine the maximum or minimum value of the function.
   a) $y = x^2 + 2x - 8$
   b) $y = -x^2 + 4x + 5$
Lesson 18-2 Homework

Lesson Summary/Reflection:

#1-2 For the function \( y = -x^2 + x + 6 \), use the value of \(-\frac{b}{2a}\) to respond to the following.
1. Identify the vertex of the graph.  
2. Write the equation of the axis of symmetry.

#3-6 Use the value of \(-\frac{b}{2a}\) to determine the vertex and to write the equation for the axis of symmetry for the graph of each of the following quadratic functions.
3. \( y = x^2 - 10x \)  
4. \( y = x^2 - 4x - 32 \)

5. \( y = x^2 + x - 12 \)  
6. \( y = 6x - x^2 \)
7. Describe the graph of a quadratic function that has its vertex and a zero at the same point.

**Model with mathematics.** An architect is designing a tunnel and is considering using the function \( y = -0.12x^2 + 2.4x \) to determine the shape of the tunnel’s entrance, as shown in the figure. In this model, \( y \) is the height of the entrance in feet and \( x \) is the distance in feet from one end of the entrance.

a) How wide is the tunnel’s entrance at its base?

b) What is the vertex? What does it represent?

c) Could a truck that is 14 feet tall pass through the tunnel? Explain.
Lesson 18-3: Graphing a Quadratic Function

Objectives:
- Use the axis of symmetry, the vertex, and the zeros to graph a quadratic function.
- Interpret the graph of a quadratic function.

NOTES:
If a quadratic function can be written in factored form, you can graph it by finding the vertex and the zeros.

Example A Graph the quadratic function \( y = x^2 - x - 12 \).
Determine the axis of symmetry, vertex, and zeros.

Try These A

a. Check the graph in Example A by plotting two more points on the graph. First, choose an \( x \)-value, and then find the \( y \)-value by evaluating the function. Plot this point. Then plot the reflection of the point over the axis of symmetry to get another point. Verify that both points are on the graph.

Graph the quadratic functions by finding the vertex and the zeros. Check your graphs.

b. \( y = x^2 - 2x - 8 \)  
c. \( y = 4x - x^2 \)  
d. \( y = x^2 + 4x - 5 \)
Joseph, Carter, and Alisha tested a new trebuchet designed to launch the projectile even further. They also refined their model to reflect a more accurate launch height of 1 m. The new projectile path is given by the function \( y = -\frac{1}{19}(x^2 - 18x - 19) \).

1. Graph the projectile path on the coordinate axes below.

2. Make sense of problems. Last year’s winning trebuchet launched a projectile a horizontal distance of 19.5 m. How does the team’s trebuchet compare to last year’s winner?
Lesson 18-3 Homework

Lesson Summary/Reflection:

#1-7 Graph the quadratic functions. Label the vertex, axis of symmetry, and zeros on each graph.

1. \( y = x^2 - 11x + 30 \)

2. \( y = x^2 - 9 \)

3. \( y = x^2 + 5x \)
4. $y = -x^2 + 2x$

5. $y = -x^2 + \frac{1}{4}$

6. $y = x^2 - 3x - 4$

7. $y = -x^2 + 3x + 4$

8. **Attend to precision.** Describe the similarities and differences between the graphs of the functions in Item 6 and 7. How are these similarities and differences indicated by the functions themselves?
LESSON 18 PRACTICE
Additional practice problems from lessons 18-1, 18-2, and 18-3.

Lesson 18-1

#1-3 Use the graphs to determine the zeros of the quadratic functions.

1. \( y = 0.5x^2 \)  
2. \( y = x^2 - 4 \)  
3. \( y = -x^2 + 4x - 3 \)

4. Identify the zeros of the quadratic function \( y = (x - a)(x - b) \).

5. Which of the following functions has zeros at \( x = 2 \) and \( x = -3 \)?
   A. \( y = x^2 + x - 6 \)  
   B. \( y = -x^2 + 5x + 6 \)  
   C. \( y = 2x^2 + x - 3 \)  
   D. \( y = x^2 + 2x - 3 \)

6. What are the solutions to the equation \( x^2 - 6x = -5 \)?
   A. \( -1, 5 \)  
   B. \( 2, 3 \)  
   C. \( 1, 5 \)  
   D. \( -2, 3 \)

#7-13 Solve by factoring.

7. \( 0 = x^2 - 25 \)  
8. \( 0 = x^2 + 17x \)  
9. \( 0 = x^2 - 5x - 66 \)
10. $0 = x^2 + 99x - 100$  
11. $0 = -x^2 - 4x + 32$  
12. $0 = x^2 + 10x + 25$  

13. $0 = x^2 + 8x + 7$  

14. A fountain at a city park shoots a stream of water vertically from the ground. The function $y = -8x^2 + 16x$ models the height of the stream of water in feet after $x$ seconds.  
   a) What is the maximum height of the stream of water?  
   b) At what time does the stream of water reach its maximum height?  
   c) For how many seconds does the stream of water appear above ground?  
   d) Identify a reasonable domain and range for this function.

15. Write a quadratic equation whose roots are $-4$ and $8$. How do the roots relate to the zeros and factors of the associated quadratic function?  

16. Write a quadratic function whose zeros are $2$ and $9$. How do the zeros relate to the factors of the quadratic function?
Lesson 18-2

17. If a quadratic function has zeros at $x = -4$ and $x = 6$, what is the $x$-coordinate of the vertex?

18. What is the vertex of the quadratic function $y = x^2 + 8x + 15$?

19. Write the equation for the axis of symmetry of the graph of the quadratic function $y = 2x^2 + 4x - 1$.

Use the following information for Items 20-24.

A diver in Acapulco jumps from a cliff. His height $y$, in meters, as a function of $x$, his distance from the cliff base in meters, is given by the quadratic function $y = 100 - x^2$, for $x \geq 0$.

20. Graph the function representing the cliff diver’s height. 
   a) Identify the vertex of the graph.

   b) Identify the $x$-intercepts of the graph.

21. Determine the solutions of the equation $100 - x^2 = 0$.

22. How do the solutions to the equation in Item 21 relate to the $x$-intercepts of the graph in Item 20?

23. How high is the cliff from which the diver jumps?

24. How far from the base of the cliff does the diver hit the water?
25. Which of these functions has a graph with the axis of symmetry \( x = -2 \)?
   
   A. \( y = x^2 + 4x - 2 \)  
   B. \( y = x^2 - 4x + 2 \)  
   C. \( y = 2x^2 + 2x - 3 \)  
   D. \( y = 2x^2 - 2x + 3 \)

26. Lisa correctly graphs a quadratic function and found that its vertex was in Quadrant I. Which function could she have graphed?
   
   A. \( y = x^2 + 4x + 2 \)  
   B. \( y = x^2 - 4x + 2 \)  
   C. \( y = x^2 - 4x + 6 \)  
   D. \( y = x^2 + 4x + 6 \)

27. Which of these is the equation of the parabola graphed on the right?
   
   A. \( y = x^2 + 4x + 2 \)  
   B. \( y = x^2 - 4x + 2 \)  
   C. \( y = x^2 - 4x + 6 \)  
   D. \( y = x^2 + 4x + 6 \)

28. Deshawn’s textbook shows the graph of the function \( y = x^2 + x - 6 \). Which of these is a true statement about the graph?
   
   A. The axis of symmetry is the \( y \)-axis.  
   B. The vertex is \((0, -6)\).  
   C. The graph intersects the \( x \)-axis at \((-3, 0)\).  
   D. The graph is a parabola that opens downward.

29. Kim throws her basketball up from the ground toward the basketball hoop from a distance of 20 feet away from the hoop. The ball follows a parabolic path and returns back to the gym floor 5 feet from the hoop. Write one possible equation to represent the path of the basketball. Explain your answer.

MATHEMATICAL PRACTICES: Use Appropriate Tools Strategically

30. Consider the quadratic function \( y = x^2 - x - 3 \).
   
   a) Is it possible to find the zeros of the function by factoring?
   
   b) Use your calculator to graph the function. Based on the graph, what are the approximate zeros of the function?
   
   c) Use the zero function of your calculator to find more accurate approximations for the zeros. Round to the nearest tenth.
Common Core State Standards for Lesson 19
HSA-SSE.B.3: Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.
HSA-SSE.B.3b: Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.
HSA-REI.B.4: Solve quadratic equations in one variable.
HSA-REI.B.4a: Use the method of completing the square to transform any quadratic equation in x into an equation of the form \((x - p)^2 = q\) that has the same solutions. Derive the quadratic formula from this form.
HSA-REI.B.4b: Solve quadratic equations by inspection (e.g., for \(x^2 = 49\)), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as \(a \pm bi\) for real numbers \(a\) and \(b\).
HSF-IF.C.8: Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
HSF-IF.C.8a: Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

Lesson 19-1: The Square Root Method
Objectives:
- Solve quadratic equations by the square root method
- Provide examples of quadratic equations having a given number of real solutions.

NOTES: Nguyen is trying to build a square deck around his new hot tub. To decide how large a deck he should build, he needs to determine the side length, \(x\), of different sized decks given the possible area of each deck. He knows that the area of a square is equal to the length of a side squared. Using this information, Nguyen writes the following equations to represent each of the decks he is considering.

1. Solve each equation. Be prepared to discuss your solution methods with your classmates.
   a. \(x^2 = 49\)  
   b. \(x^2 = 15\)  
   c. \(2x^2 = 18\)  
   d. \(x^2 - 4 = 0\)  
   e. \(x^2 + 2 = 0\)  
   f. \(x^2 + 3 = 3\)  

2. Refer to the equations in Item 1 and their solutions.
   a. What do the equations have in common?
   b. What types of numbers are represented by the solutions of these equations?
c. How many solutions do the equations have?

d. **Reason quantitatively.** Which solutions are reasonable for side lengths of the squares? Explain.

To solve a quadratic equation of the form $ax^2 + c = 0$, isolate the $x^2$-term and then take the square root of both sides.

**Example A**  Solve $3x^2 - 6 = 0$ using square roots.

**Try These A** Solve each equation using square roots.

a. $x^2 - 10 = 1$  
  b. $\frac{x^2}{4} = 1$  
  c. $4x^2 - 6 = 14$

3. Quadratic equations can have 0, 1, or 2 real solutions. Fill in the table below with equations from the first page that represent the possible numbers of solutions.

<table>
<thead>
<tr>
<th>Number of Solutions</th>
<th>Result When $x^2$ is Isolated</th>
<th>Example(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two</td>
<td>$x^2 = \text{positive number}$</td>
<td></td>
</tr>
<tr>
<td>One</td>
<td>$x^2 = 0$</td>
<td></td>
</tr>
<tr>
<td>No real solutions</td>
<td>$x^2 = \text{negative number}$</td>
<td></td>
</tr>
</tbody>
</table>
4. Reason abstractly. A square frame has a 2 in. border along two sides as shown in the diagram. The total area is $66in^2$. Answer the questions to help you write an equation to find the area of the unshaded square.

![Diagram of a square frame with a 2 in. border along two sides]

a. Label the sides of the unshaded square $x$.
b. Fill in the boxes to write an equation for the total area in terms of $x$.

\[
\text{Area in terms of } x = \text{Area in square in.}
\]

You can solve quadratic equations like the one you wrote in Item 4 by isolating the variable.

Example B Solve $(x + 2)^2 = 66$ using square roots. Approximate the solutions to the nearest hundredth.

5. Are both solutions to this equation valid in the context of Item 4? Explain your response.

Try These B Solve each equation using square roots.
a. $(x - 5)^2 = 121$  
b. $(2x - 1)^2 = 6$  
c. $x^2 - 12x + 36 = 2$
Lesson 19-1 Homework

Lesson Summary/Reflection:

Solve each equation using square roots.
1. $x^2 + 12 = 13$ 
2. $(x - 4)^2 = 1$ 
3. $3x^2 - 6 = 15$

3. Give an example of a quadratic equation that has
a. One real solution. 
   b. No real solutions. 
   c. Two real solutions.

4. If the length of a square is decreased by 1 unit, the area will be 8 square units. Write an equation for the area of the square.

5. Calculate the side length of the square in Item 4.

Solve each equation
6. $x^2 - 22 = 0$ 
7. $(x + 5)^2 - 4 = 0$

8. $x^2 - 4x + 4 = 0$ 
9. $(x + 1)^2 = 12$
10. **Model with mathematics.** Alisha has a square picture with an area of 100 square inches, including the frame. The width of the frame is $x$ inches.

   a. Write an equation in terms of $x$ for the area $A$ of the picture inside the frame.

   b. If the area of the picture inside the frame is 64 square inches, what are the possible values for $x$?

   c. If the area of the picture inside the frame is 64 square inches, how wide is the picture frame? Justify your response.
Lesson 19-2: Completing the Square
Objectives:
• Solve quadratic equations by completing the square.
• Complete the square to analyze a quadratic function.

NOTES:
As shown in Example B in Lesson 19-1, quadratic equations are more easily solved with square roots when the side with the variable is a perfect square. When a quadratic equation is written in the form \( x^2 + bx + c = 0 \), you can complete the square to transform the equation into one that can be solved using square roots. **Completing the square** is the process of adding a term to the variable side of a quadratic equation to transform it into a perfect square trinomial.

**Example A** Solve \( x^2 + 10x - 6 = 0 \) by completing the square.

**Try These A** Solve each quadratic equation by completing the square.

| a. \( x^2 - 8x + 3 = 11 \) | b. \( x^2 + 7 = 2x + 8 \) |

Completing the square is useful to help analyze specific features of quadratic functions, such as the maximum or minimum value and the possible number of zeros.

When a quadratic equation is written in the form \( y = a(x - h)^2 + k \), you can determine whether the function has a maximum or minimum value based on \( a \) and what that value is based on \( k \).

This information can also help you determine the number of \( x \)-intercepts.

**Example B** Analyze the quadratic function \( y = x^2 - 6x + 13 \) by completing the square.
Try These B Write each of the following quadratic functions in the form $y = a(x - h)^2 + k$. Identify the direction of opening, vertex, maximum or minimum value, and number of $x$-intercepts.

a. $y = x^2 + 8x + 15$  

b. $y = -x^2 - 6x - 8$
Lesson Summary/Reflection:

Solve by completing the square.
1. \(x^2 + 2x + 3 = 0\)
2. \(x^2 + 6x + 4 = 0\)

3. \(2 = x^2 - 10x\)
4. \(4x = x^2 - 4x - 32\)

5. \(-2x^2 + 4 = -x^2 + x - 7\)
6. \(x + 1 = 6x - x^2\)

Complete the square to determine the vertex and maximum or minimum value. Determine the number of \(x\)-intercepts.
7. \(y = x^2 - 2x + 2\)
8. \(y = -x^2 + 8x - 6\)
9. **Make sense of problems.** In a model railroad, the track is supported by an arch that is represented by \( y = -x^2 + 10x - 16 \), where \( y \) represents the height of the arch in inches and \( x \) represents the distance in inches from a cliff. Complete the square to answer the following questions.

a. How far is the center of the arch from the cliff?

b. What is the maximum height of the arch?

10. The bubbler is the part of a drinking fountain that produces a stream of water. The water in a drinking fountain follows a path given by \( y = -x^2 + 6x + 4.5 \), where \( y \) is the height of the water in centimeters above the basin, and \( x \) is the distance of the water from the bubbler. What is the maximum height of the water above the basin?
Lesson 19-3: The Quadratic Formula

Objectives:
- Derive the quadratic formula
- Solve quadratic equations using the quadratic formula.

**NOTES:**
Generalizing a solution method into a formula provides an efficient way to perform complicated procedures. You can complete the square on the general form of a quadratic equation $ax^2 + bx + c = 0$ to find a formula for solving all quadratic equations.

Solve $ax^2 + bx + c = 0$

---

**Quadratic Formula**
When $a \neq 0$, the solutions of $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To apply the quadratic formula, make sure the equation is in standard form $ax^2 + bx + c = 0$. Identify the values of $a$, $b$, and $c$ in the equation and then substitute these values into the quadratic formula. If the expression under the radical sign is not a perfect square, write the solutions in simplest radical form or use a calculator to approximate the solutions.

**Example A** Solve $x^2 + 3 = 6x$ using the quadratic formula.

---

**Try These A** Solve using the quadratic formula.

a. $3x^2 = 4x + 3$  
b. $x^2 + 4x = -2$
Lesson 19-3 Homework

Lesson Summary/Reflection:

Solve using the quadratic formula.

1. $3x^2 - 5x + 1 = 0$
2. $x^2 + 6 = -8x + 12$

3. $-2x^2 - x + 4 = 0$
4. $3x^2 = -6x + 4$

5. $4x^2 - 5x - 2 = 1$
6. $x^2 + 3x = -x + 1$

8. A baseball player tosses a ball straight up into the air. The function $y = -16x^2 + 30x + 5$ models the motion of the ball, where $x$ is the time in seconds and $y$ is the height of the ball, in feet.

a. Write an equation you can solve to find out when the ball is at a height of 15 feet.

b. Use the quadratic formula to solve the equation. Round to the nearest tenth.
c. How many solutions did you find for Part (b)? Explain why this makes sense.

9. Critique the reasoning of others. José and Marta each solved \( x^2 + 4x = -3 \) using two different methods. Who is correct and what is the error in the other student’s work?

<table>
<thead>
<tr>
<th>José</th>
<th>Marta</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 + 4x = -3 )</td>
<td>( x^2 + 4x = -3 )</td>
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<tr>
<td>( x^2 + 4x - 3 = 0 )</td>
<td>( x^2 + 4x + \boxed{4} = -3 + \boxed{4} )</td>
</tr>
<tr>
<td>( a = 1, b = 4, c = -3 )</td>
<td>( x^2 + 4x + 4 = -3 + 4 )</td>
</tr>
<tr>
<td>( x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-3)}}{2(1)} )</td>
<td>( (x + 2)^2 = 1 )</td>
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<td></td>
<td>( x + 2 = \pm\sqrt{1} )</td>
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<td>( x + 2 = 1 ) or ( x + 2 = -1 )</td>
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<td>( x = -1 ) or ( x = -3 )</td>
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<td>( x = \frac{-4 \pm \sqrt{16 + 12}}{2} )</td>
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<td>( x = \frac{-4 \pm \sqrt{28}}{2} )</td>
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<td>( x = \frac{-4 \pm 2\sqrt{7}}{2} = -2 \pm \sqrt{7} )</td>
</tr>
</tbody>
</table>
Lesson 19-4: Choosing a Method and Using the Discriminant

Objectives:
- Choose a method to solve a quadratic equation.
- Use the discriminant to determine the number of real solutions of a quadratic equation.

NOTES:
There are several methods for solving a quadratic equation. They include factoring, using square roots, completing the square, and using the quadratic formula. Each of these techniques has different advantages and disadvantages. Learning how and why to use each method is an important skill.

1. Solve each equation below using a different method. State the method used.
   a. \( x^2 + 5x - 24 = 0 \)  
   b. \( x^2 - 6x + 2 = 0 \)
   c. \( 2x^2 + 3x - 5 = 0 \)  
   d. \( x^2 - 100 = 0 \)

2. How did you decide which method to use for each equation in Item 1?

The expression \( \sqrt{b^2 - 4ac} \) in the quadratic formula helps you understand the nature of the quadratic equation. The **discriminant**, \( b^2 - 4ac \), of a quadratic equation gives information about the number of real solutions, as well as the number of \( x \)-intercepts of the related quadratic function.
3. Solve each equation using any appropriate solution method. Then complete the rest of the table.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Discriminant</th>
<th>Solutions</th>
<th>Number of Real Solutions</th>
<th>Number of x-Intercepts</th>
<th>Graph of Related Quadratic Function</th>
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</thead>
<tbody>
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<td>$x^2 + 2x - 3 = 0$</td>
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<td></td>
<td><img src="image1.png" alt="Graph" /></td>
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<tr>
<td>$x^2 + 2x + 1 = 0$</td>
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<td><img src="image2.png" alt="Graph" /></td>
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<tr>
<td>$x^2 + 2x + 5 = 0$</td>
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<td><img src="image3.png" alt="Graph" /></td>
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</tbody>
</table>

4. **Express regularity in repeated reasoning.** Complete each statement below using the information from the table in Item 3.

- If $b^2 - 4ac > 0$, the equation has ________ real solution(s) and the graph of the related function has ________ x-intercept(s).
- If $b^2 - 4ac = 0$, the equation has ________ real solution(s) and the graph of the related function has ________ x-intercept(s).
- If $b^2 - 4ac < 0$, the equation has ________ real solution(s) and the graph of the related function has ________ x-intercept(s).
Lesson Summary/Reflection:

Use the discriminant to determine the number of real solutions.

1. $4x^2 + 2x - 12 = 0$
2. $x^2 - 10x + 25 = 0$

For each equation, use the discriminant to determine the number of real solutions. Then solve the equation.

3. $x^2 - 1 = 0$
4. $x^2 - 4x + 4 = 0$

5. $-4x^2 + 3x = -2$
6. $x^2 - 2 = 12x$

7. $x^2 + 5x - 1 = 0$
8. $-x^2 - 2x - 10 = 0$
9. **Model with mathematics.** Lin launches a model rocket that follows a path given by the function $y = -0.4t^2 + 3t + 0.5$, where $y$ is the height in meters and $t$ is the time in seconds.

a. Explain how you can write an equation and then use the discriminant to determine whether Lin’s rocket ever reaches a height of 5 meters.

b. If Lin’s rocket reaches a height of 5 meters, at approximately what time(s) does it do so? If not, what is the rocket’s maximum height?
Lesson 19-5: Complex Solutions
Objectives:
- Use the imaginary $i$ to write complex numbers.
- Solve a quadratic equation that has complex solutions.

NOTES:
When solving quadratic equations, there are always one, two, or no real solutions. Graphically, the number of $x$-intercepts is helpful for determining the number of real solutions.
- When there is one real solution, the graph of the related quadratic function touches the $x$-axis once, and the vertex of the parabola is on the $x$-axis.
- When there are two real solutions, the graph crosses the $x$-axis twice.
- When there are no real solutions, the graph never crosses the $x$-axis.

1. Graph the function $y = x^2 - 6x + 13$. Use the graph to determine the number of real solutions to the equation $x^2 - 6x + 13 = 0$.

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<td>-8</td>
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</table>

$x$  

2. **Construct viable arguments.** What does the number of real solutions to the equation in Item 1 indicate about the value of the discriminant of the equation? Explain.

When the value of the discriminant is less than zero, there are no real solutions. This is different from stating that there are no solutions. In cases where the discriminant is negative, there are two solutions that are not real numbers. **Imaginary numbers** offer a way to determine these non-real solutions. The **imaginary unit**, $i$, equals $\sqrt{-1}$. Imaginary numbers are used to represent square roots of negative numbers, such as $\sqrt{-4}$.  

47
Example A Simplify $\sqrt{-4}$.

Try These A Simplify.

a. $\sqrt{-16}$

b. $-\sqrt{9}$

c. $\sqrt{-8}$

Problems involving imaginary numbers can also result in complex numbers, $a + bi$, where $a$ and $b$ are real numbers. In this form, $a$ is the real part and $b$ is the imaginary part.

Example B Solve $x^2 - 6x + 12 = 0$.

Try These B Solve each equation.

a. $x^2 + 100 = 0$

b. $x^2 - 4x = -11$
Lesson Summary/Reflection:

Simplify
1. $\sqrt{-27}$
2. $-\sqrt{-11}$
3. $\sqrt{-42}$
4. $\pm\sqrt{-81}$

Solve.
5. $x^2 + 4x + 6 = 0$
6. $5x^2 - 2x + 3 = 0$
7. $-x^2 - 6 = 0$
8. $(x - 1)^2 + 3 = 0$

9. **Make use of structure.** Consider the quadratic function $y = x^2 + 2x + c$, where $c$ is a real number.
   a. Write and simplify an expression for the discriminant.
b. Explain how you can use your result from Part (a) to write and solve an inequality that tells you when the function will have two zeros that involve imaginary numbers.

c. Use your results to describe the zeros of the function $y = x^2 + 2x + 3$. 
LESSON 19 PRACTICE

Lesson 19-1
Solve each equation using square roots.

1. \( x^2 + 7 = 43 \)
2. \( (x - 5)^2 + 2 = 11 \)
3. \( x^2 - 8x + 16 = 3 \)

4. Antonio drops a rock from a cliff that is 400 feet high. The function \( y = -16t^2 + 400 \) gives the height of the rock in feet after \( t \) seconds. Write and solve an equation to determine how long it takes the rock to land at the base of the cliff. (Hint: At the base of the cliff, the height \( y \) is 0.)

5. Maya wants to use square roots to solve the equation \( x^2 - 6x + 9 = k \), where \( k \) is a positive real number. Which of these is the best representation of the solution?
   A. \( x = 3 \pm \sqrt{k} \)
   B. \( x = -3 \pm \sqrt{k} \)
   C. \( x = \pm \sqrt{k + 3} \)
   D. \( x = \pm \sqrt{k - 3} \)

Lesson 19-2
6. Given the equation \( x^2 - 8x = 3 \), what number should be added to both sides to complete the square?
   A. \(-4\)  
   B. \(8\)  
   C. \(16\)  
   D. \(64\)

Write each of the following equations in the form \( y = a(x - h)^2 + k \). Then identify the direction of opening, vertex, maximum or minimum value, and \( x \)-intercepts.

7. \( y = x^2 - 4x + 11 \)
8. \( y = -x^2 - 6x - 8 \)
9. \( y = x^2 + 2x - 8 \)

10. A golfer stands on a platform 16 feet above a driving range. Once the golf ball is hit, the function \( y = -16t^2 + 64t + 16 \) represents the height of the ball in feet after \( t \) seconds.
   a. Write an equation you can solve to determine the number of seconds it takes for the ball to land on the driving range.

   b. Solve the equation by completing the square. Leave your answer in radical form.

   c. Use a calculator to find the number of seconds, to the nearest tenth, that it takes the ball to land on the driving range.

11. Which of the following is a true statement about the graph of the quadratic function \( y = x^2 - 2x + 3 \)?
   A. The vertex of the graph is \((-1, 2)\).
   B. The graph intersects the x-axis at \( x = 1 \).
   C. The graph is a parabola that opens upward.
   D. There is exactly one x-intercept.
Solve by completing the square.
12. \( x^2 - 4x = 12 \) \hspace{1cm} 13. \( x^2 + 10x + 21 = 0 \)

14. \( 2x^2 - 4x - 4 = 0 \) \hspace{1cm} 15. \( x^2 + 6x = -10 \)

16. A climbing structure at a playground is represented by the function \( y = -x^2 + 4x + 1 \), where \( y \) is the height of the structure in feet and \( x \) is the distance in feet from a wall. What is the maximum height of the structure?
   A. 1 foot \hspace{1cm} B. 2 feet \hspace{1cm} C. 4 feet \hspace{1cm} D. 5 feet

**Lesson 19-3**
Solve using the quadratic formula.
17. \( 4x^2 - 4x = 3 \) \hspace{1cm} 18. \( 5x^2 - 9x - 2 = 0 \)

19. \( x^2 = 2x + 4 \)
20. A football player kicks a ball. The function \( y = -16t^2 + 32t + 3 \) models the motion of the ball, where \( t \) is the time in seconds and \( y \) is the height of the ball in feet.

a. Write an equation you can solve to find out when the ball is at a height of 11 feet.

b. Use the quadratic formula to solve the equation. Round to the nearest tenth.

21. Kyla was asked to solve the equation \( 2x^2 + 6x - 1 = 0 \). Her work is shown below. Is her solution correct? If not, describe the error and give the correct solution.

\[
2x^2 + 6x - 1 = 0 \\
a = 2, \ b = 6, \ c = -1
\]

\[
x = \frac{-6 \pm \sqrt{6^2 - 4(2)(-1)}}{2(2)}
\]

\[
= \frac{-6 \pm \sqrt{36 + 8}}{4}
\]

\[
= \frac{-6 \pm \sqrt{44}}{4}
\]

\[
= \frac{-6 \pm 2\sqrt{11}}{4}
\]

\[
= \frac{-3 \pm \sqrt{11}}{2}
\]

**Lesson 19-4**

Use the discriminant to determine the number of real solutions.

22. \( x^2 + 3x + 5 = 0 \)

23. \( 4x^2 - 4x + 1 = 0 \)
24. The discriminant of a quadratic equation is \(-1\). Which of the following must be a true statement about the graph of the related quadratic function?
A. The graph intersects the x-axis in exactly two points.
B. The graph lies entirely above the x-axis.
C. The graph intersects the x-axis at \(x = -1\).
D. The graph has no x-intercepts.

25. A dolphin jumps straight up from the water. The quadratic function \(y = -16t^2 + 20t\) models the motion of the dolphin, where \(t\) is the time in seconds and \(y\) is the height of the dolphin, in feet. Use the discriminant to explain why the dolphin does not reach a height of 7 feet.

**Lesson 19-5**

Simplify.

26. \(\pm \sqrt{-2}\)  
27. \(-\sqrt{-25}\)  
28. \(\sqrt{-8}\)

29. \(-\sqrt{-121}\)  
30. \(12 - \sqrt{-144}\)  
31. \(\pm \sqrt{-32}\)

Solve.

32. \(2x^2 - 5x + 5 = 0\)  
33. \(x^2 + x + 3 = 0\)
34. \(-3x^2 - 3x - 1 = 0\)  
35. \(-x^2 - x - 2 = 0\)

36. For what values of \(p\) does the quadratic function \(y = x^2 + 4x + p\) have two real zeros? Justify your answer.
Lesson 20-1: Solving a System Graphically

Objectives:
- Write a function to model a real-world situation.
- Solve a system of equations by graphing.

NOTES:

Professor Hearst is studying different types of bacteria in order to determine new ways to prevent their population overgrowth. Each bacterium in the first culture that she examines divides to produce another bacterium once each minute. In the second culture, she observes that the number of bacteria increases by 10 bacteria each second.

1. Each population began with 10 bacteria. Complete the table for the population of each bacteria sample.

<table>
<thead>
<tr>
<th>Elapsed Time (minutes)</th>
<th>Population of Sample A</th>
<th>Population of Sample B</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Describe the type of function that would best model each population.

3. Write a function $A(t)$ to model the number of bacteria present in Sample A after $t$ minutes.

4. Write a function $B(t)$ to model the number of bacteria present in Sample B after $t$ minutes.

5. Use a graphing calculator to graph $A(t)$ and $B(t)$ on the same coordinate plane.
   a. Sketch the graph and label several points on each graph.
   b. Determine the points of intersection of the two graphs. Round non-integer values to the nearest tenth.
c. **Make use of structure.** What do the points of intersection indicate about the two graphs? Explain.

d. Interpret the meaning of the points of intersection within the context of the bacteria samples.


The solutions you found in Item 5b are solutions to the **nonlinear system of equations** \( A(t) = 10(2)^t \) and \( B(t) = 600t + 10 \).

Just as with linear systems, you can solve nonlinear systems by graphing each equation and determining the intersection point(s).

7. Solve each system of equations by graphing.

   a. \[
   \begin{align*}
   y &= 2x + 1 \\
   y &= x^2 + 1
   \end{align*}
   \]

   b. \[
   \begin{align*}
   y &= x - 1 \\
   y &= 3^x - 2
   \end{align*}
   \]
c. \[ y = 2(5)^x \]
   \[ y = -x^2 + 2x - 2 \]

8. a. Examine the graphs in #7. How many solutions are possible for a nonlinear system of linear, quadratic, and/or exponential equations?

b. Describe how this is different from the number of possible solutions for a linear system.
Lesson 20-1 Homework

Lesson Summary/Reflection:

1. A population of bacteria is given by \( f(t) = t^2 + 2t + 30 \), where \( t \) is in minutes. Another population begins with 10 bacteria and doubles every minute – it can be modeled by the function \( g(t) = 10(2)^t \). Graph the two functions and determine the time the two populations are equal.

#2-4 Solve each system of equations by graphing.

2. \[
\begin{align*}
  y &= -2x + 4 \\
  y &= -x^2 + 3
\end{align*}
\]

3. \[
\begin{align*}
  y &= x^2 + 5x - 3 \\
  y &= -x^2 + 5x + 1
\end{align*}
\]
4. \( y = -4x^2 - 1 \)
   \( y = 4(0.5)^x \)

5. A nonlinear system contains one linear equation and one quadratic equation. The system has no solution. Sketch a possible graph of this system in the grid above (next to #4).

6. Is it possible for a system with one linear equation and one exponential equation to have two solutions? If so sketch a graph that could represent such a system. If not explain why not.

7. **Critique the reasoning of others.** A population of 200 bacteria begins increasing at a constant rate of 100 bacteria per minute. Francis writes the function \( P(t) = 200(100)^t \), where \( t \) represents the time in minutes, to model this population. Fred disagrees. He writes the function \( P(t) = 200 + 100t \) to model this population. Who is correct? Justify your response.
Lesson 20-2: Solving a System Algebraically

Objectives:
- Write a system of equations to model a real-world situation.
- Solve a system of equations by algebraically.

NOTES:
Just as with linear systems, nonlinear systems of equations can be solved algebraically.

Example A Solve the system of equations algebraically.
\[ y = -x + 3 \]
\[ y = x^2 - 2x - 3 \]

Try These A Solve the system of equations algebraically.

a) \[ y = -x + 2 \]
   \[ y = x^2 - x + 2 \]

b) \[ y = 2x^2 - 7 \]
   \[ y = 7x - 3 \]

c) \[ y = -x + 3 \]
   \[ y = x^2 - 2x - 4 \]

1. Lauren solved the following system of equations algebraically and found two solutions.
   \[ y = -x + 3 \]
   \[ y = x^2 - 2x - 4 \]
Will solved the system by graphing and said that there is only one solution. Who is correct? Justify your response both algebraically and graphically.
2. Model with mathematics. Deshawn drops a ball from the 520-foot high observation deck of a tower. The height of the ball in feet after $t$ seconds is given by $f(t) = -16t^2 + 520$. At the moment the ball is dropped, Zoe begins traveling up the tower in an elevator that starts at the ground floor. The elevator travels at a rate of 12 feet per second. At what time will Zoe and the ball pass by each other?

a. Write a function $g(t)$ to model Zoe’s height above the ground after $t$ seconds.

b/c. Write a system of equations using the function modeling the height of the ball and the function you wrote in Part (a). Solve the system. Round to the nearest hundredth, if necessary.

d. Interpret the meaning of the solution in the context of the problem. Does the solution you found make sense? Explain.

e. Determine the height at which Zoe and the ball pass by each other. Explain how you found your answer.

3. At the same moment that Deshawn drops the ball, Joey begins traveling down the tower in another elevator that starts at the observation deck. This elevator also travels at a rate of 12 feet per second.

a. Write a function $h(t)$ to model Joey’s height above the ground after $t$ seconds. Explain any similarities or differences between this function and the function in Item 2a.

b/c. Solve the system of equations algebraically. Interpret the solution in the context of the problem.

d. Construct viable arguments. Determine whether Joey or the ball reaches the ground first. Justify your response.
Lesson 20-2 Homework

Lesson Summary/Reflection:

1. How many solutions could the following system of equations have?
   \[ y = x + 8 \]
   \[ y = x^2 - 10x + 8 \]

2. Solve the system of equations in #1 using any appropriate algebraic method.

#3-5 Solve each system algebraically.

3. \( y = 16x - 13 \)
   \[ y = 4x^2 + 3 \]

4. \( y = 5 \)
   \[ y = -x^2 - x + 1 \]

5. \( y = x \)
   \[ y = x^2 + 2x - 4 \]

6. Jessica has decided to solve a system of equations by graphing. Her graph is shown. Why might she prefer to solve this system algebraically?
LESSON 20 PRACTICE
Additional practice problems from lessons 20-1 and 20-2.

Lesson 20-1

1. Which function models the size of a neighborhood that begins with one home and doubles in size every year?
   A. $P(t) = t + 2$   B. $P(t) = 2t + 1$   C. $P(t) = 2(1)^t$   D. $P(t) = (2)^t$

2. Which function models the size of a neighborhood that begins with 4 homes and increases by 6 homes every year?
   A. $P(t) = 4t + 6$   B. $P(t) = 6t + 4$   C. $P(t) = 6^t + 4$   D. $P(t) = 4(6)^t$

3. When will the number of homes in #1 and #2 be equal?

For #4-5, sketch the graph of a system that matches the description. If no such system exists, write not possible.

4. The system contains a linear equation and an exponential equation. There are no solutions.

5. The system contains two exponential equations. There is one solution.

For #6-7, solve each system of equations by graphing.

6. $y = -x^2 + 3$
   $y = x^2 + 4$

7. $y = 2x^2 + 5$
   $y = -2x + 5$
For #8-9, write a system of equations to model the situation. Then solve the system by graphing.

8. A sample of bacteria starts with 2 bacteria and doubles every minute. Another sample starts with 4 bacteria and increases at a constant rate of 2 bacteria every minute. When will the populations be equal?

9. Jennie and James plan to save money by raking leaves. Jennie already has 1 penny. With each bag of leaves she rakes, she doubles the amount of money she has. James earns 10 cents per bag. When will Jennie have more money than James?

Lesson 20-2

For #10-14, solve each system of equations algebraically.

10. \[ y = 3x^2 - x - 2 \]
    \[ y = 2x + 3 \]

11. \[ y = x^2 - 81 \]
    \[ y = 18x - 161 \]
12. \( y = x^2 + 4 \)  
\( y = 4x \)

13. \( y = 3x + 3 \)  
\( y = x^2 + 3x + 2 \)

14. \( y = 3x \)  
\( y = 2x^2 \)

15. Write a system of equations to model the scenario. Then solve the system algebraically. Simone is driving at a rate of 60 mi/h on the highway. She passes Jethro just as he begins accelerating onto the highway from a complete stop. The distance that Jethro has traveled in feet after \( t \) seconds is given by the function \( f(t) = 5.5t^2 \). When will Jethro catch up to Simone? (Hint: Use the fact that 60 mi/h is equivalent to 88 ft/s to write a function that gives the distance Simone has traveled)

16. **Construct Viable Arguments and Critique the Reasoning of Others**
Rachel is solving systems of equations and has concluded that the quadratic formula is always an appropriate solution method when solving a nonlinear system algebraically. Do you agree with Rachel’s conclusion? Use examples to support your reasoning.
Common Core State Standards for Lesson 21
HSS-ID.A.1: Represent data with plots on the real number line (dot plots, histograms, and box plots).
HSS-ID.A.2: Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (IQR, standard deviation) of two or more different data sets.
HSS-ID.A.3: Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).
HSS-ID.A.4: Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.

Lesson 21-1: Mean, Median, Mode, and MAD
Objectives:
- Interpret differences in center and spread of data.
- Compare center and spread of two or more data sets.
- Determine the mean absolute deviation of a data set.

NOTES:
Zach is a high school student who enjoys texting with friends after school. Recently, Zach’s parents have become concerned about the amount of time that he spends text messaging on school nights.

Zach decides to compare the amount of time he spends text messaging to that of his good friend Olivia. Both of them record the number of minutes they spend text messaging on school nights for one week.

One way to describe a set of data is by explaining how the data cluster around a value, or its center. The measures of center include the mean, the median, and the mode.  
1. Find the mean amount of time that Zach spends text messaging each night. Show how you determined your answer.

<table>
<thead>
<tr>
<th></th>
<th>Sunday</th>
<th>Monday</th>
<th>Tuesday</th>
<th>Wednesday</th>
<th>Thursday</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zach</td>
<td>10 min</td>
<td>60 min</td>
<td>20 min</td>
<td>135 min</td>
<td>75 min</td>
</tr>
<tr>
<td>Olivia</td>
<td>60 min</td>
<td>60 min</td>
<td>60 min</td>
<td>60 min</td>
<td>60 min</td>
</tr>
</tbody>
</table>

2. Find the mean amount of time that Olivia spends text messaging each night. Show how you determined your answer.

3. Reason quantitatively. Compare the amounts of time that Zach and Olivia spend text messaging. Describe similarities and differences.

Zach knows that data can be described by center and also by spread. Spread indicates how far apart the data values are in the set. Measures of spread include the range and the mean absolute deviation.

Zach asks his friend Trey to record the amount of time he spends text messaging on school nights. To measure spread, Zach chooses the range.
4. Find the mean and range of Trey’s data.

5. Complete the table below. How do the mean and range of Trey’s data compare to those of Zach’s?

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trey</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Zach</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

6. Construct viable arguments. Describe how the two data sets are different. Did the mean and range help you to identify these differences? Explain.

Because the range is based on only two values, it does not reflect any variation in the data between the greatest and least values. The range is greatly influenced by extreme values. Another measure of spread that is not as influenced by extremes is the mean absolute deviation, which is computed using all the data values. The **mean absolute deviation** is the mean (average) of the absolute values of the deviations of the data. The **deviation** is a measure of how far a data value is from the mean.

7. To find the mean absolute deviation of Zach’s data, begin by completing the table. Use the mean for Zach's data that you calculated in Item 1.

8. To finish calculating the mean absolute deviation of Zach’s data, find the mean of the numbers in the third column. Determine the sum of the numbers in the third column and then divide by the number of data values (the number of items in the first column).

<table>
<thead>
<tr>
<th>Zach</th>
<th>Deviation</th>
<th>Absolute Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>Time — Mean = $(x - \bar{x})$</td>
<td>$</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>60</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>135</td>
<td></td>
<td></td>
</tr>
<tr>
<td>75</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

9. Reason abstractly. Why would statisticians use the mean absolute deviation rather than the mean of the deviations (in the second column)?

10. Trey’s text messaging minutes are shown in the table above.
   a. Find the mean absolute deviation for the amount of time that Trey spends text messaging.
b. Why is the mean absolute deviation for Trey’s data set greater than the mean absolute deviation for Zach’s data set?

11. Olivia’s text messaging minutes are also given in the table above Item 10.
   a. Find the mean absolute deviation for the amount of time that Olivia spends text messaging.

b. Make sense of problems. Explain why Olivia’s mean absolute deviation is descriptive of her data.
Lesson 21-1 Homework

Lesson Summary/Reflection:

1. During the annual food drive, Mr. Binford’s homeroom collected canned goods for a month. The numbers of cans collected are given below.

<table>
<thead>
<tr>
<th>Boys</th>
<th>Girls</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>20</td>
</tr>
<tr>
<td>42</td>
<td>63</td>
</tr>
<tr>
<td>69</td>
<td>18</td>
</tr>
<tr>
<td>91</td>
<td>89</td>
</tr>
<tr>
<td>97</td>
<td>67</td>
</tr>
<tr>
<td>61</td>
<td>19</td>
</tr>
<tr>
<td>15</td>
<td>66</td>
</tr>
<tr>
<td>37</td>
<td>108</td>
</tr>
<tr>
<td>104</td>
<td>96</td>
</tr>
<tr>
<td>38</td>
<td>24</td>
</tr>
<tr>
<td>82</td>
<td>16</td>
</tr>
<tr>
<td>90</td>
<td>44</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Boys</th>
<th>Girls</th>
</tr>
</thead>
<tbody>
<tr>
<td>51</td>
<td>66</td>
</tr>
<tr>
<td>96</td>
<td>108</td>
</tr>
<tr>
<td>19</td>
<td>96</td>
</tr>
<tr>
<td>66</td>
<td>24</td>
</tr>
<tr>
<td>8</td>
<td>16</td>
</tr>
<tr>
<td>24</td>
<td>44</td>
</tr>
</tbody>
</table>

   Compare and contrast the results for the boys and girls using the mean and range.

   __________________________________________________________
   __________________________________________________________

2. Remi recorded data on her car’s fuel efficiency for five trips in the table below.

<table>
<thead>
<tr>
<th>Trip</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Miles per gallon</td>
<td>23.7</td>
<td>25.5</td>
<td>25.2</td>
<td>24.8</td>
<td>25.4</td>
</tr>
</tbody>
</table>

   a. Calculate the absolute deviation for each trip if the average number of miles per gallon for the trips was 24.9.

   __________________________________________________________

   b. Find the mean absolute deviation for the five trips. ________________

3. Vernice asked 12 classmates to record the number of hours they spent watching television during one week. The table shows the data she collected.

   | Hours | 10 | 11 | 22 | 17 | 17 | 20 | 31 | 0  | 12 | 19 | 23 |

   a. Calculate the mean and mean absolute deviation for the data.

   Mean: _____    MAD ________

   b. What statements could Vernice make about the viewing habits of these classmates?

   __________________________________________________________
4. Scores from the same benchmark test were collected from two algebra classes, each with 30 students enrolled. One class had a mean score of 79 with a mean absolute deviation of 5, and the other had a mean score of 81 with a mean absolute deviation of 10. What can be said about the distribution of scores on this test for the two classes?


5. Mitch and a group of his friends have estimated how long it will take each of them to run 400 meters around the track. Their estimates in seconds are 115, 76, 96, 81, 78, 99, 68, and 84.

a. Calculate the mean and the range. ______________________________

b. Which estimate stands out as unusual? ______________________________

c. What might be a reason for such an unusual estimate? ______________________________
Lesson 21-2: Dot Plots and Box Plots

Objectives:
- Construct representations of univariate data in a real-world context.
- Describe characteristics of the data distribution, such as center, shape, and spread, using graphs and numerical summaries.
- Compare distributions, commenting on similarities and differences among them.

NOTES:

Professional wildlife managers and the public are concerned with the impact of human activity on wildlife. One measure studied is animals’ “home range,” the typical area in which an animal spends its time. Researchers were concerned that the home ranges of some coyotes in a portion of Colorado were affected by military maneuvers involving jeeps, tanks, helicopters, and jet fighter flyovers. To evaluate these potential effects, several coyotes were collared with radio transmitters. The researchers used the transmitters to track the movement of the coyotes. Coyotes were monitored before, during, and after the military maneuvers.

A dot plot is an effective method for representing univariate (one-variable) data when dealing with small data sets.

1. A dot plot of the “Before” data is shown. Make dot plots of the “During” and “After” data sets.

2. Compare and contrast the centers and spreads of the three data sets.

A five-number summary provides a numerical summary of a set of data. It is used to construct a box plot. (The first quartile ($Q_1$) is the median of the data values to the left of the overall median, and the third quartile ($Q_3$) is the median of the data values to the right of the overall median.) The five-number summary for the “Before” home ranges is shown here, together with the resulting box plot.
3. Create five-number summaries of the “During” and “After” data.

<table>
<thead>
<tr>
<th>Home Ranges: During Maneuvers</th>
<th>Home Ranges: After Maneuvers</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Minimum:</strong></td>
<td><strong>Minimum:</strong></td>
</tr>
<tr>
<td><strong>First quartile:</strong></td>
<td><strong>First quartile:</strong></td>
</tr>
<tr>
<td><strong>Median:</strong></td>
<td><strong>Median:</strong></td>
</tr>
<tr>
<td><strong>Third quartile:</strong></td>
<td><strong>Third quartile:</strong></td>
</tr>
<tr>
<td><strong>Maximum:</strong></td>
<td><strong>Maximum:</strong></td>
</tr>
</tbody>
</table>

4. Use the summaries in Item 3 to construct box plots of the “During” and “After” data sets.

![Before box plot]

During

![After box plot]

After

5. Based on the box plots and five-number summaries in Items 3 and 4:

a. Which data set seems to have the least overall spread? Which data set seems to have the greatest overall spread?

b. Which data set seems to have the least spread in its “middle 50%” box? Which data set seems to have the greatest spread in its “middle 50%” box?

Remember that the initial concern before the data gathering was that the home ranges of the local coyotes might change during the military maneuvers. To investigate this concern, you will use the graphs, numerical summaries, and comparisons you have developed as a starting point for your analysis.

6. Based on the data and graphs, does it appear that there was a substantial change in the coyotes’ home ranges during the military maneuvers? Write a few sentences specifically comparing the “Before” and “During” data sets. Use numerical values where possible.
7. Do there appear to be any substantial permanent changes to the coyotes’ home ranges after the military maneuvers? Write a few sentences specifically comparing the “Before” and “After” data sets. Use numerical values where possible.

8. The “During” data set contains two values that are far away from the rest of the data. The “Before” data set contains one such value also. Suppose you wish to call attention to the fact there are such far-away data values. Which type of plot—the dot plot or the box plot—would be your choice? Why?
Lesson 21-2 Homework

Lesson Summary/Reflection:

1. A teacher in a statistics class allows her students to use notes about statistical procedures on tests. She believes that a teacher-made study sheet will be more effective in helping students recall the procedures. In each of her three classes she used one of three helping strategies (a) student-made notes with information about the procedures, (b) teacher-made information printed on paper in the form of a flowchart, and (c) teacher-made information delivered by computer access during the exam. Each of her classes has 18 students. The test scores for her students are given as percent correct.

| Student Notes | 89 | 15 | 39 | 15 | 31 | 69 | 39 | 54 | 31 | 62 | 46 | 39 | 54 | 39 | 15 | 46 | 23 | 31 |
| Paper Help    | 76 | 24 | 77 | 71 | 18 | 29 | 59 | 77 | 41 | 77 | 77 | 47 | 71 | 82 | 82 | 82 | 59 | 65 |
| Computer Help | 100| 13 | 73 | 73 | 33 | 53 | 60 | 60 | 27 | 80 | 80 | 47 | 73 | 80 | 80 | 93 | 60 | 53 |

a. In order to compare the results for these three groups, construct a dot plot for each of the three data sets.

b. Describe the three data sets with specific attention to center and spread.
2. For each group, create a five-number summary for these data.

<table>
<thead>
<tr>
<th></th>
<th>Student Notes</th>
<th>Paper Help</th>
<th>Computer Help</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimum</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>First Quartile</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Median</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Third Quartile</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Maximum</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. Use the summaries in #2 to construct box plots for the three data sets: “Notes,” “Paper,” and “Computer.”

4. a. For these data, what advantages do you see in using the dot plots to display the data sets?

   b. For these data, what advantages do you see in using the box plots to display the data sets?

5. Comparing the test scores for these groups and using specific information from the five-number summary and/or your dot plots and box plots, answer the following in a few sentences.

   a. Which group appears to have done the best on the exam?

   b. Which group appears to have done the worst on the exam?
Lesson 21-3: Modified Box Plots

Objectives:
- Use modified box plots to summarize data in a way that shows outliers.
- Compare distributions, commenting on similarities and differences among them.

**NOTES:**
You already are familiar with many ways to summarize data graphically and numerically. In this lesson, you will see a new type of plot called a **modified box plot**.

Here are the dot plots of the data about the coyotes from Lesson 21-2. Refer back to Lesson 21-2 for the actual data values. Notice that there are two unusually large values in the “During” data set and one in the “Before” data set.

**Outliers** are values that differ so much from the rest of a one-variable data set that attention is drawn to them. Outliers may arise for many reasons, including measurement errors or recording errors. It may also be the case that there actually are unusual values in a data set.

Outliers can also occur in bivariate (two-variable) data. It is important to consider the impact of outliers when summarizing and analyzing data.

1. Which, if any, of the data values shown in the dot plots do you consider to be outliers? List each data value that you think might be an outlier and the data set from which it came.

2. Describe the method you used in Item 1 to determine whether a data value is an outlier. Is it based on distance? Is it based on the concentration of other data values? Compare your method with those of your classmates.

A data value is considered to be an outlier if it is more than $1.5(IQR)$ from the nearest quartile. Recall that $IQR$ stands for **interquartile range** and is the difference between the third quartile ($Q_3$) and the first quartile ($Q_1$): $IQR = Q_3 - Q_1$.

We will use this definition of an outlier to draw a **modified** box plot. The idea behind modifying the box plot is to create a plot that shows outliers.

When creating a modified box plot, the whiskers extend only to the least data value that is not an outlier and to the greatest data value that is not an outlier. Outliers are then shown by adding dots to the box plot to indicate their locations.
Let’s walk through the steps together using the data from the “Before” data set. Here are the data, arranged in order, as well as the five-number summary and box plot of the data from Lesson 21-2.

| 3.5 | 3.6 | 3.9 | 4.8 | 5.3 | 5.3 | 5.4 | 5.5 | 5.7 | 6.3 | 11.4 |

The maximum value, 11.4, is much greater than the other data values. In fact, it is more than 5 units away from the next-greatest value of 6.3. The remaining data (from the minimum value of 3.5 to 6.3) have a range of only 2.8.

Now let’s modify the box plot to show outliers.

3. For the “Before” data set, how great or how small must a data value be to be identified as an outlier? Perform the calculations below to find the upper and lower boundary values that separate outliers from the rest of the data.

\[ Q_3 + 1.5(IQR) = \text{___________} = \text{____} \]

\[ Q_1 - 1.5(IQR) = \text{___________} = \text{____} \]

4. Are any data values less than the lower boundary for outliers identified in Item 3?

5. Are any data values greater than the upper boundary for outliers identified in Item 3?

Next, identify the least and greatest values in the data set that are not outliers. These values will determine the endpoints of the whiskers in the modified box plot.

6. What is the least data value that is not an outlier?
7. What is the greatest data value that is not an outlier?

Now you have everything you need to draw the modified box plot. The modified box plot is constructed as follows:

**Step 1.** Draw the box as usual.
**Step 2.** Extend the whiskers to the least and greatest data values that are not outliers.
**Step 3.** Place dots above the scale to indicate the outliers.

8. Follow the directions above to draw the modified box plot for the “Before” data:
Lesson 21-3 Homework

Lesson Summary/Reflection:

1. A data set has a third quartile of 64 and a first quartile of 29. What are the upper and lower boundary values that separate outliers from the rest of the data set?

2. **Make sense of problems.** If the third quartile of the data set in #1 were increased by 10, how would this change the upper boundary for outliers? Explain your answer.

3. **Critique the reasoning of others.** In a survey of 21 teenage girls about their text message usage for one month, the five-number summary is minimum = 0, Q1 = 1, median = 31, Q3 = 56, and maximum = 1305. In analyzing these data, Michelle determined that there are no outliers. Do you agree or disagree? Explain.

4. Describe the effects an outlier can have on a set of data.

5. **Construct viable arguments.** In a data set, the third quartile is 36 and the first quartile is 12. Would a value of 52 be considered an outlier? Why or why not?

6. Lisa recorded the heights of her classmates. Her data are shown in the table below:

<table>
<thead>
<tr>
<th>Heights of Classmates in Inches</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>61</td>
<td>58</td>
<td>62</td>
<td>60</td>
<td>57</td>
</tr>
<tr>
<td>67</td>
<td>68</td>
<td>61</td>
<td>64</td>
<td>70</td>
</tr>
<tr>
<td>72</td>
<td>64</td>
<td>63</td>
<td>59</td>
<td>69</td>
</tr>
</tbody>
</table>
a. Calculate the upper and lower boundaries for outliers for the heights of Lisa’s classmates.

b. List two values that would be considered outliers in this data set. Include one value that is less than the lower boundary for outliers and one that is greater than the upper boundary for outliers.
LESSON 21 PRACTICE
Additional practice problems from lessons 21-1, 21-2, and 21-3.

Lesson 21-1
1. Use the data set \{2, 12, 6, 7, 3, 10, 6, 2, 10, 8\} to find the following:

   a. Mean
   b. Median
   c. Mode
   d. Mean Absolute Deviation (MAD)

2. The amount of caffeine in beverages presents an important health concern, especially for women of childbearing age. In a recent study of carbonated sodas, the numbers of milligrams of caffeine detected were as follows:

   \[29.5, 38.2, 39.6, 29.5, 31.7, 27.4, 45.4, 48.2, 36.0, 33.8, 19.4, 18.0, 34.6\]

   Calculate the mean, median, mode, and mean absolute deviation for these data.

3. In a report of the caffeine levels, five sodas were left out because no caffeine was detected. If these sodas were given a value of 0mg of caffeine and added to the data above, how would the mean, median, mode and mean absolute deviation change?

Lesson 21-2

A restaurant specializing in Mexican food offers nine different choices of enchiladas. They vary in cost and in sodium content. Data for the nine options are given in the table below.

<table>
<thead>
<tr>
<th>Option</th>
<th>Cost/Serving (dollars)</th>
<th>Sodium Content (mg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.03</td>
<td>780</td>
</tr>
<tr>
<td>2</td>
<td>1.07</td>
<td>1570</td>
</tr>
<tr>
<td>3</td>
<td>1.28</td>
<td>1500</td>
</tr>
<tr>
<td>4</td>
<td>1.53</td>
<td>1370</td>
</tr>
<tr>
<td>5</td>
<td>1.05</td>
<td>1700</td>
</tr>
<tr>
<td>6</td>
<td>1.27</td>
<td>1330</td>
</tr>
<tr>
<td>7</td>
<td>2.34</td>
<td>440</td>
</tr>
<tr>
<td>8</td>
<td>2.47</td>
<td>520</td>
</tr>
<tr>
<td>9</td>
<td>2.09</td>
<td>660</td>
</tr>
</tbody>
</table>

4. A dot plot and a box plot of the Cost/Serving amounts are shown below. What feature of the
data set is apparent in the box plot but not particularly apparent in the dot plot?

5. A dot plot and a box plot of the Sodium Content amounts are shown below. What feature of the data set is hidden in the box plot but apparent in the dot plot?

6. In a study of raptors in the western United States, 110 Cooper’s hawks were trapped and their weights (in grams) recorded. A dot plot of these weights is shown below. What interesting feature do you notice about this data set? What do you think might be the reason for this interesting feature?

For winter sports enthusiasts, the thickness of ice is a significant safety issue. The Minnesota Department of Natural Resources recommends that ice thickness be at least 4 inches for walking or skating on the ice, and at least 5 inches for operating a snowmobile or all-terrain vehicle on the ice. Ice thickness (in inches) were measured at 10 randomly selected locations on the surface of a lake. The thicknesses were as follows:

5.8, 6.4, 6.9, 7.2, 5.1, 4.9, 4.3, 5.8, 7.0, 6.8
7. Construct a dot plot of the ice thicknesses.

8. On the basis of your dot plot, do you think it is safe to play hockey on this lake? Explain why or why not.

9. On the basis of your dot plot, do you think it is safe to operate a snowmobile on this lake? Explain why or why not.

10. Calculate the mean ice thickness for the locations in this sample.

11. Calculate the mean average deviation of the ice thickness.

12. If the mean of the thicknesses were greater and the mean average deviation were the same, would you be more worried or less worried about operating a snowmobile on the ice on this lake? Explain.

13. If the mean of the thicknesses were the same and the mean average deviation were greater, would you be more worried or less worried about walking or skating on the ice on this lake? Explain.

**Lesson 21-3**

A regional symphony orchestra needs money to repair their theater, which was seriously damaged by flooding. They have tested three different methods of asking for donations: mail, phone, and direct appeal at social gatherings. Each method was used with 11 potential donors, and the amounts donated for each method are shown on the next page. Use the table for #14-16.
14. Complete the table below:

<table>
<thead>
<tr>
<th>Minimum</th>
<th>Mail</th>
<th>Phone</th>
<th>Direct</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Quartile</td>
<td>1100</td>
<td>1500</td>
<td>1000</td>
</tr>
<tr>
<td>Median</td>
<td>1300</td>
<td>1750</td>
<td>1200</td>
</tr>
<tr>
<td>Third Quartile</td>
<td>1500</td>
<td>1850</td>
<td>1350</td>
</tr>
<tr>
<td>Maximum</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lower Outlier Boundary</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Upper Outlier Boundary</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

15. Construct modified box plots for the different methods.

16. Based on the data and your box plots, which method would you recommend and why?
17. Sometimes it is not clear whether a box plot is a modified box plot or a standard box plot. If you were looking at a box plot and outliers were not visible, what characteristics of the plot would lead you to believe it was standard rather than modified?

The following values represent the number of states visited by students in a class:

3, 12, 17, 2, 21, 14, 14, 8, 45, 29

18. Find the interquartile range and any outliers for the data set.

19. If you found an outlier in #18, what does this number represent? Does it make sense that this number would be an outlier in this context?

20. Create a modified box plot for the data.

21. Two new students joined the class, both of whom have visited only two states each. What effects, if any, does this have on the upper and lower boundaries for outliers?

22. **Use Appropriate Tools Strategically.** In #4 of the Lesson 21-2 homework, you were asked about advantages of using box plots and dot plots to describe and compare distributions of scores. Do you think the advantages you found would exist not only for these data, but for numerical data in general? Explain.
Common Core State Standards for Lesson 22
HSS-ID.C.8 Compute (using technology) and interpret the correlation coefficient of a linear fit.
HSS-ID.C.9 Distinguish between correlation and causation.
HSS-ID.B.6 Represent data on two quantitative variables on a scatter plot, and describe how variables are related:
   a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. Use given
      functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential
      models.
   b. Informally assess the fit of a function by plotting and analyzing residuals.
   c. Fit a linear function for a scatter plot that suggests a linear association.

Lesson 22-1: Scatter Plots
Objectives:
- Describe a linear relationship between numerical variables in terms of direction and
  strength.
- Use the correlation coefficient to describe strength and direction of a linear relationship
  between two numerical variables.

NOTES:
Scatter plots are used to visualize the relationship between two numerical variables. When you look at a scatter plot, determine
whether there appears to be a relationship (pattern) between the two variables.

For Scatter Plot 1, there does appear to be a relationship between $x$ and $y$ because greater values of $x$ tend to be paired with greater
values of $y$. Notice that the pattern in the plot looks roughly linear,
so you would say that there is a linear relationship between these
two variables.

1. For Scatter Plot 2, does there appear to be a relationship between $x$ and $y$? If so, describe the pattern.

2. For Scatter Plot 3, does there appear to be a relationship between $x$ and $y$? If so, describe the pattern.

There is a summary statistic to describe the strength (how close the points are to a line) and
direction (positive or negative) of a linear relationship. This statistic is called the correlation
coefficient and is denoted by $r$. 
3. Given below are seven scatter plots and seven verbal descriptions of relationships. Match each scatter plot with the appropriate description. (Each scatter plot goes with one and only one description.)

A. Very strong positive linear relationship \( (r = 0.981) \)  

B. Relatively strong positive linear relationship \( (r = 0.828) \)  

C. Relatively weak positive linear relationship \( (r = 0.310) \)  

D. Very slight or no linear relationship \( (r = 0.043) \)  

E. Relatively weak negative linear relationship \( (r = -0.238) \)  

F. Relatively strong negative linear relationship \( (r = 0.772) \)  

G. Very strong negative linear relationship \( (r = -0.95) \)  

4. What feature(s) of the scatter plots did you consider when deciding whether a relationship was positive or negative?

5. What feature(s) of the scatter plots did you consider when deciding whether a relationship was relatively weak, relatively strong, or very strong?

6. Make sense of problems. Examine the values of \( r \) for each relationship in Item 3. How does the value of \( r \) relate to the scatter plots? What makes \( r \) increase or decrease?

Here is a summary of important characteristics of \( r \):
- The value of \( r \) quantifies the strength of a linear relationship.
- The sign of \( r \) describes the direction of the relationship: positive or negative.
- \( r \) ranges in value between \(-1\) (perfect negative linear relationship) and \(+1\) (perfect positive linear relationship).
Lesson 22-1 Homework

Lesson Summary/Reflection:

The following table displays costs to travel, round-trip, to various cities from Cedar Rapids, Iowa. The costs are calculated assuming a June 1 departure and a 3-day stay. Driving costs were calculated based on $0.20 per mile.

<table>
<thead>
<tr>
<th>Destination</th>
<th>Train</th>
<th>Plane</th>
<th>Car</th>
</tr>
</thead>
<tbody>
<tr>
<td>New York City</td>
<td>268</td>
<td>391</td>
<td>204</td>
</tr>
<tr>
<td>Chicago</td>
<td>74</td>
<td>453</td>
<td>49</td>
</tr>
<tr>
<td>Atlanta</td>
<td>483</td>
<td>703</td>
<td>168</td>
</tr>
<tr>
<td>Washington, D.C.</td>
<td>254</td>
<td>577</td>
<td>186</td>
</tr>
<tr>
<td>New Orleans</td>
<td>338</td>
<td>342</td>
<td>189</td>
</tr>
<tr>
<td>Denver</td>
<td>221</td>
<td>384</td>
<td>160</td>
</tr>
<tr>
<td>Albuquerque</td>
<td>354</td>
<td>486</td>
<td>222</td>
</tr>
<tr>
<td>Seattle</td>
<td>510</td>
<td>647</td>
<td>367</td>
</tr>
<tr>
<td>San Francisco</td>
<td>290</td>
<td>435</td>
<td>385</td>
</tr>
<tr>
<td>Los Angeles</td>
<td>390</td>
<td>299</td>
<td>362</td>
</tr>
<tr>
<td>Kansas City</td>
<td>184</td>
<td>523</td>
<td>64</td>
</tr>
</tbody>
</table>

Scatter plots of cost versus distance for each of the three travel methods are shown below.

1. **Reason abstractly.** How would you describe the relationship between cost and distance for each method of transportation? Be sure to indicate whether you think the relationship is linear and to comment on the strength and direction of the relationship.
   
a) Train  
b) Plane  
c) Car

90
2. Describe the relationship shown in the scatter plot.

3. **Reason abstractly.** In your own words, describe the similarities and differences between a scatter plot that shows a strong positive relationship and a scatter plot that shows a weak positive relationship.

4. What type of relationship would you expect to see between height and age? Explain your answer.

5. Describe two real-world quantities that would have a strong negative relationship.

6. Describe two real-world quantities that would have no correlation.

7. For positive linear relationships, as the value of \( r \) increases, is the linear relationship getting stronger or weaker?
Lesson 22-2: Correlation Coefficient

Objectives:
- Calculate correlation.
- Distinguish between correlation and causation.

NOTES:
The calculation of $r$ gives additional information that helps to describe the data.

This data set shows price (in dollars) and quality ratings for 12 different brands of bike helmets. The quality rating is a number from 0 (worst) to 100 (best) that measures various factors such as how well the helmet absorbed the force of an impact, the strength and ventilation of the helmet, and its ease of use.

<table>
<thead>
<tr>
<th>Bicycle Helmets</th>
<th>Price (dollars)</th>
<th>Quality Rating</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>35</td>
<td>20</td>
</tr>
</tbody>
</table>

![Graph showing correlation between price and quality rating]

1. a) How would you describe the relationship between price and quality rating?

b) Make a prediction of what you think the correlation coefficient might be.

2. Using a graphing calculator, enter the prices as one list and the quality ratings as another list. What is the value of the correlation coefficient for these two variables?

3. Reason abstractly. How would you interpret the value of the correlation coefficient in the context of this problem?
At this point you have used scatter plots to visually represent the relationship between two numerical variables and you have used a numerical measure to describe the strength and direction of a linear relationship. When a relationship is uncovered by statistics, the next task is to explain its meaning.

Sometimes the interpretation of a relationship may not be obvious. For example, across European countries there is a positive linear relationship between the number of storks and the number of newborn babies. Do storks bring babies? Are storks attracted by babies? Are both babies and storks brought by the tooth fairy? Do parents with newborns have warmer houses, and therefore their chimneys attract storks looking for warm places to nest?

Humans want to make sense of their world, and sometimes leap too quickly from seeing a correlation to inferring causation (a cause-and-effect relationship between two variables). This tendency should be resisted! There are many reasons why two variables might be related other than cause and effect. Here are some common examples where a correlation should not be interpreted as a cause-and-effect relationship:

- The number of fire engines responding to a fire is positively correlated with the total damage. (Should fewer fire engines — perhaps 0 — be sent to fires to reduce damage?)

- The number of people drowning at beaches is positively correlated with ice cream sales. (Is ice cream dangerous?)

- Shoe size is strongly correlated with reading ability. (Should parents start their children off with size 12?)

- The number of doctors per 1000 people is positively correlated with the rate of serious disease. (Are doctors spreading disease?)

4. Make sense of problems. For each of the correlations above, what do you think is the correct explanation for the correlation?
Lesson 22-2 Homework

Lesson Summary/Reflection:

1. For each of the following pairs of variables, indicate whether you would expect a positive correlation, a negative correlation, or a correlation close to 0. Explain your choice.
   
a) Weight of a car and gas mileage

b) Size and selling price of a house

c) Height and weight

d) Height and number of siblings

The table gives data on age and number of cell phone calls made in a typical day for each person in a random sample of 10 people. Use the table for Items 2-4.

2. Sketch a scatter plot of these data using the grid below.

3. Describe the direction and strength of the relationship between these two variables.

4. Calculate the value of the correlation coefficient. Do the sign and the magnitude of the correlation coefficient agree with your answer in Item 3? Explain.

5. Suppose that you shot an arrow into the air and kept track of how high it was every 1.0 second. If you made a scatter plot of the data (time, height), the resulting pattern of points would be in the shape of a parabola. Do you feel the correlation coefficient should be used to describe the strength of the relationship between time and height? Why or why not?
**Model with mathematics.** Consider table computers with 9- to 10-inch screen.

6. For tablet computers, do you think there is a relationship between price and battery life? If so, do you think the relationship is positive or negative?

7. For tablet computers, do you think there is a relationship between price and weight? If so, do you think the relationship is positive or negative?

Data for tablet computers with 9- to 10-inch screens are shown in the table and scatter plots below.

<table>
<thead>
<tr>
<th>Price (dollars)</th>
<th>Battery Life (hours)</th>
<th>Weight (pounds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>730</td>
<td>11.6</td>
<td>1.3</td>
</tr>
<tr>
<td>570</td>
<td>8.4</td>
<td>1.0</td>
</tr>
<tr>
<td>600</td>
<td>11.6</td>
<td>1.3</td>
</tr>
<tr>
<td>600</td>
<td>9.3</td>
<td>1.2</td>
</tr>
<tr>
<td>600</td>
<td>9.1</td>
<td>1.3</td>
</tr>
<tr>
<td>800</td>
<td>8.9</td>
<td>1.3</td>
</tr>
<tr>
<td>600</td>
<td>10.5</td>
<td>1.6</td>
</tr>
<tr>
<td>850</td>
<td>11.5</td>
<td>1.6</td>
</tr>
<tr>
<td>500</td>
<td>11.0</td>
<td>1.6</td>
</tr>
<tr>
<td>470</td>
<td>9.0</td>
<td>1.5</td>
</tr>
<tr>
<td>500</td>
<td>8.6</td>
<td>1.4</td>
</tr>
<tr>
<td>500</td>
<td>8.4</td>
<td>1.6</td>
</tr>
<tr>
<td>480</td>
<td>7.4</td>
<td>1.7</td>
</tr>
<tr>
<td>500</td>
<td>8.6</td>
<td>1.7</td>
</tr>
<tr>
<td>570</td>
<td>7.7</td>
<td>1.7</td>
</tr>
<tr>
<td>780</td>
<td>8.1</td>
<td>1.7</td>
</tr>
<tr>
<td>580</td>
<td>9.5</td>
<td>2.1</td>
</tr>
</tbody>
</table>

8. Calculate the correlation coefficient for Price and Battery Life.

9. Calculate the correlation coefficient for Price and Weight.

10. Are the values of the correlation coefficients consistent with your predictions in Items 6 and 7? Explain.
Lesson 22-3: Line of Best Fit

Objectives:
- Describe the linear relationship between two numerical variables using the best-fit line.
- Use the equation of the best-fit line to make predictions and compare them to actual values.

NOTES:
In recent activities, you created scatter plots as a way to graphically summarize bivariate numerical data. In addition, you learned how to use the correlation coefficient as a numerical summary of the strength and direction of a linear relationship.

In this activity, you will see a way to summarize bivariate numerical data called the “best-fit line.” You will use technology to determine the slope and y-intercept of the best-fit line for a data set.

1. The scatter plots show linear relationships of different strengths and directions. For each scatter plot, use your judgment to draw a line that you feel best represents the linear relationship.

2. Compare the lines you drew with the lines drawn by another student in your class. Did you draw identical lines? Were your lines more similar for scatter plots where the linear relationship was strong or where the linear relationship was weak?

Because informal assessments of what line might best describe a linear relationship don’t always agree, we need to come to some agreement about what “best” means.

Before we look at how to define the best-fit line, let’s first consider how the best-fit line might be used.

One reason for finding a best-fit line to describe the relationship between two variables is so that you can use the line to make predictions. For example, you might want to predict the age (in
years) of a black bear from its weight (in pounds). This would be helpful to wildlife biologists, because it is a lot easier to weigh a bear than to ask a bear its age!

Suppose you know that for adult black bears, the relationship between age and weight can be approximately described by the line

\[ y = -3.69 + 0.115x \]

where \( y \) = age in years and \( x \) = weight in pounds. You can use this equation to predict the age of a bear that weighs 100 pounds:

\[
\text{predicted age} = -3.69 + 0.115(100) = -3.69 + 11.5 = 7.81 \text{ years}
\]

3. Using the equation \( y = -3.69 + 0.115x \), what is the predicted age of a bear that weighs 115 pounds?

The line \( y = -3.69 + 0.115x \) is the best-fit line for the following data. These data are from a study in which nine black bears of known age were weighed.

4. Construct a scatter plot for the bear data.

5. Add the best-fit line to your scatter plot.

(Hint: Find two points on the line by picking two \( x \) values and using the equation of the best-fit line to find the corresponding predicted ages. Then plot these two \( (x, \text{predicted age}) \) pairs and draw a line that goes through those two points.)

Below is a scatter plot of the bear data, the best-fit line, and a line that is not the best-fit line.
6. Why is the best-fit line a better description of the relationship between age and weight than the other line graphed?

7. One bear in the data set (Bear 3) was 5.5 years old and weighed 92.6 pounds. If you used the best-fit line \( y = -3.69 + 0.115x \) to predict the age of this bear based on its weight, how far off would you be from the bear’s actual age?

8. If you used the line graphed on the previous page to predict the age of this bear, do you think your prediction would be closer to or further from the bear’s actual age? What feature(s) of the scatter plot shown above supports your answer?

9. **Attend to precision.** For the bear that was 5.5 years old and weighed 92.6 pounds, the best-fit line led to a predicted age that was greater than the bear’s actual age. Will age predictions based on the best-fit line be greater than the actual age for all of the bears in the data set? If so, explain why. If not, give an example of a bear in the data set for which the predicted age is less than the bear’s actual age.
Lesson Summary/Reflection:

1. The scatter plot shows two lines, Line 1 and Line 2. One of these lines is the best-fit line. Which one is it?

2. Suppose that for students taking a statistics class, the best-fit line for a data set where y is a student’s test score (out of 100 points) and x is the number of hours spent studying for the test is \( y = 43 + 12x \).
   
a) What is the predicted test scores for a student who studied for one hour?
   
b) What is the predicted test score for a student who studied for three hours?

Mr. Trent examined some data on head height and a person’s actual height and found that a person’s height is about 7.5 times his or her head height. “Head height” refers to the distance from the top of the head to the bottom of the chin. Using the data he gathered, Mr. Trent found that the equation of the best-fit line is \( y = 2.5 + 7.5x \), where y represents height in inches and x represents head height in inches. Use this equation for Items 3-5.

3. Xavier’s head height is 8.3 inches. Predict Xavier’s height.

4. Tamisha is 5 foot 3 inches tall. Predict her head height.

5. **Reason quantitatively.** Tori said that she is 64 inches tall and her head height is 8 inches. Is this possible? Explain.
Lesson 22-4: Residuals

Objectives:
- Use technology to determine the equation of the best-fit line.
- Describe the linear relationship between two numerical variables using the best-fit line.
- Use residuals to investigate whether a given line is an appropriate model of the relationship between numerical variables.

NOTES:
Here is a scatter plot of the bear data with the best-fit line and the points in the scatter plot labeled according to which bear the data point represents.

1. For which bears does the best-fit line predict an age that is less than the bear’s actual age?

2. Look at the points in the scatter plot that correspond to the bears whose predicted ages are less than their actual ages. What do the points all have in common relative to the best-fit line?

For Bear 3, the actual age was 5.5 years and the predicted age from the best-fit line was 6.96 years. The difference between the actual age and the predicted age is $5.5 - 6.96 = -1.46$

3. Look at the scatter plot and locate the point corresponding to Bear 3. What does 1.46 represent in terms of the scatter plot?

The difference between an actual $y$-value and a predicted $y$-value is called a residual. A residual is positive when the actual $y$-value is greater than the predicted $y$-value.

4. When is a residual negative?

5. For which of the bears is the residual positive?

6. Look at the scatter plot. Do data points that fall above the best-fit line have positive or negative residuals?
The table shows the actual ages, predicted ages using the best-fit line, and residuals for the nine bears.

**7. Make sense of problems.** What is the sum of all nine residuals? Does this value surprise you? Explain why or why not.

<table>
<thead>
<tr>
<th>Bear</th>
<th>Age (years)</th>
<th>Predicted Age (years)</th>
<th>Residual</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.5</td>
<td>6.45</td>
<td>0.05</td>
</tr>
<tr>
<td>2</td>
<td>7.5</td>
<td>6.45</td>
<td>1.05</td>
</tr>
<tr>
<td>3</td>
<td>5.5</td>
<td>6.96</td>
<td>−1.46</td>
</tr>
<tr>
<td>4</td>
<td>8.0</td>
<td>9.00</td>
<td>−1.00</td>
</tr>
<tr>
<td>5</td>
<td>10.5</td>
<td>9.25</td>
<td>1.25</td>
</tr>
<tr>
<td>6</td>
<td>9.5</td>
<td>9.25</td>
<td>0.25</td>
</tr>
<tr>
<td>7</td>
<td>10.5</td>
<td>10.01</td>
<td>0.49</td>
</tr>
<tr>
<td>8</td>
<td>9.0</td>
<td>10.26</td>
<td>−1.26</td>
</tr>
<tr>
<td>9</td>
<td>11.5</td>
<td>11.27</td>
<td>0.23</td>
</tr>
</tbody>
</table>

*Note:* The sum of the residuals for the best-fit line is equal to zero. Here, because of rounding in the calculation of the slope and y-intercept of the best-fit line and rounding in calculating the predicted values, the sum of the residuals is not exactly 0.

**8.** The scatter plot shows the best-fit line and another line. If you ignore the sign of the residuals, which line has greater residuals overall? *(Hint: Look at the distances of points to each of the two lines.)*

A line is a good description of a bivariate data set if the residuals tend to be small overall. To measure the overall “goodness” of a line, you might think about adding all of the residuals. The problem with this is that some residuals are positive and some are negative, and so you can get a sum that is zero (or close to zero) even for lines that are not good descriptions of the data. So, instead of judging the “goodness” of a line by looking at the sum of the residuals, we look at the sum of the squared residuals (**SSR**, for short). The squared residuals are all positive, so positive and negative values don’t offset one another.

**9.** Look again at the scatter plot above that shows the bear data and the two different lines. Which line do you think has the lesser SSR? Explain your reasoning.

The **best-fit line** for a particular data set is the line that has the least sum of squared residuals (least SSR). In the scatter plot with the two lines, not only does the best-fit line have an SSR less than that of the other line shown, it has an SSR less than that of any other line.

Calculating the equation of the best-fit line by hand is very time-consuming, especially if there are a lot of values in the data set. Because of this, you will use a graphing calculator or computer software to do the calculations.

**10.** Enter the bear data and use technology to verify that the equation of the best-fit line is $y = −3.69 + 0.115x$. 

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Lesson 22-3 Homework

Lesson Summary/Reflection:

The men’s basketball coach at Grinnell College employs a style of basketball known as “system ball.” The idea behind system ball is that forcing turnovers on defense leads to more shots, especially 3-point shots, on offense, and thus a higher point total. Data on the number of turnovers committed by the opposing team and the total points scored by Grinnell for a sample of seven games are given below.

1. Construct a scatter plot for this data set.

<table>
<thead>
<tr>
<th>Turnovers (x)</th>
<th>Total Points Scored (y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>36</td>
<td>115</td>
</tr>
<tr>
<td>45</td>
<td>126</td>
</tr>
<tr>
<td>26</td>
<td>103</td>
</tr>
<tr>
<td>18</td>
<td>106</td>
</tr>
<tr>
<td>25</td>
<td>117</td>
</tr>
<tr>
<td>31</td>
<td>128</td>
</tr>
<tr>
<td>22</td>
<td>96</td>
</tr>
</tbody>
</table>

2. Based on the scatter plot, how would you describe the relationship between \( x \) and \( y \)?

3. Use technology to find the equation of the best-fit line.

4. Use the best-fit line to predict the total points scored in a game with 30 turnovers.
The table below shows the historical minimum wage (in dollars per hour) for the State of New York. Use the tables for Items 5-8.

<table>
<thead>
<tr>
<th>Year (x)</th>
<th>Wage (y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1962</td>
<td>1.15</td>
</tr>
<tr>
<td>1968</td>
<td>1.60</td>
</tr>
<tr>
<td>1974</td>
<td>2.00</td>
</tr>
<tr>
<td>1978</td>
<td>2.65</td>
</tr>
<tr>
<td>1981</td>
<td>3.35</td>
</tr>
<tr>
<td>1990</td>
<td>3.80</td>
</tr>
<tr>
<td>1991</td>
<td>4.25</td>
</tr>
<tr>
<td>2000</td>
<td>5.15</td>
</tr>
<tr>
<td>2005</td>
<td>6.00</td>
</tr>
<tr>
<td>2012</td>
<td>7.25</td>
</tr>
</tbody>
</table>

5. Construct a scatter plot of the data.

6. Does there appear to be a relationship between the year and the minimum wage? If so, describe the relationship.

7. Use appropriate tools strategically. Find the equation for the best-fit line and the correlation coefficient.

8. Construct viable arguments. Do the equation you found and the correlation coefficient support your answer in Item 6? Explain.
LESSON 22 PRACTICE

Additional practice problems from lessons 22-1, 22-2, 22-3, and 22-4.

Lessons 22-1 and 22-2

1. Describe as precisely as you can how the appearance of a scatter plot showing a positive linear relationship between two quantitative variables differs from the appearance of a scatter plot showing a negative relationship between two quantitative variables.

2. Describe as precisely as you can how the appearance of a scatter plot showing a strong linear relationship between two quantitative variables differs from the appearance of a scatter plot showing a weak linear relationship between two quantitative variables.

The basking shark is the second-largest fish (after the whale shark) swimming in the oceans today. In a study of these creatures, their length and average swimming speed were measured from a safe distance. The results are shown in the table below. Use the table for Items 3–5.

3. Sketch a scatter plot of the data.

<table>
<thead>
<tr>
<th>Body Length (meters)</th>
<th>Average Speed (meters/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.0</td>
<td>0.89</td>
</tr>
<tr>
<td>4.5</td>
<td>0.83</td>
</tr>
<tr>
<td>4.0</td>
<td>0.76</td>
</tr>
<tr>
<td>6.5</td>
<td>0.94</td>
</tr>
<tr>
<td>5.5</td>
<td>0.94</td>
</tr>
</tbody>
</table>

4. Calculate the correlation coefficient for these data using your available technology.

5. How would you describe this relationship in terms of strength and direction? Support your description with specific references to the scatter plot and/or the correlation coefficient.
One danger of premature human birth is low birth weight. It is thought that low birth weight results in small hippocampus volume, which might be cause for concern because the hippocampus is important in later brain functioning. The scatter plot below displays data from a study of the relationship between hippocampus volume and birth weight in premature infants. The correlation coefficient for these data is $r = 0.51$.

6. Describe the strength and direction of this relationship.

7. Does this relationship appear to be reasonable described as linear? Explain.

When young children are prepared for surgery, a tracheal tube is inserted to allow the unconscious child to breathe. It is very important to get the correct insertion depth. Researchers investigated the relationship between best insertion depth and the weight of the child in a large sample of children, and a scatter plot of their data is shown. The correlation coefficient is $r = 0.878$.

8. Describe the strength and direction of this relationship.

**MATHEMATICAL PRACTICES: Reason Abstractly and Quantitatively**

9. Does this relationship appear to be reasonable described as linear? Why or why not?
Lessons 22-3 and 22-4

A veterinarian is studying the relationship between the weight of one-year-old golden retrievers in pounds \( (y) \) and the amount of dog food the dog is fed each day in pounds \( (x) \). A random sample of 10 one-year-old golden retrievers yielded the data in the table below. Use the table for Items 10–12.

<table>
<thead>
<tr>
<th>Dog</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>0.5</td>
<td>1.0</td>
<td>0.6</td>
<td>1.0</td>
<td>1.3</td>
<td>1.5</td>
<td>0.5</td>
<td>0.7</td>
<td>0.8</td>
<td>0.6</td>
</tr>
<tr>
<td>( y )</td>
<td>42</td>
<td>67</td>
<td>47</td>
<td>55</td>
<td>62</td>
<td>71</td>
<td>36</td>
<td>42</td>
<td>50</td>
<td>43</td>
</tr>
</tbody>
</table>

10. Construct a scatter plot of the dog food data.

The equation of the best fit line is \( y = 24.6 + 31.7x \); the correlation coefficient is \( r = 0.92 \).

11. One dog in this data is fed 0.8 pound of food per day. If you used the best-fit line to predict the weight of this dog, how far off would you be from the dog’s actual weight?

12. Suppose that you graphed the best-fit line on your scatter plot in Item 10. Looking at the scatter plot with the best-fit line, how would you know whether a point had a positive or a negative residual?
Extra Notes: